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Bifurcations into Pathology for Hamiltonian Systems<sup>1</sup>

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This paper presents a geometric analysis of bifurcations leading to chaos for Hamiltonian systems with two degrees of freedom of the form  $\dot{x} = y$ ,  $\dot{y} = -\nabla V(x)$ . Two bifurcation parameters are considered. One is the energy level and the other is an angle,  $\Phi$ , between two homoclinic orbits. Though global non-linearities are necessary, the results are obtained by local analysis of the flow near the origin where it is assumed that  $D^2V(0) = -I$ , the  $2 \times 2$  identity matrix. For a fixed energy level it is shown that as  $\Phi$  decreases through  $90^\circ$  the two homoclinic orbits bifurcate into two homoclinic orbits, a periodic orbit, and connecting orbits. These orbits can then be used to define a compact region in  $\mathbb{R}^4$ . Now treating the energy as a parameter value the trajectory of orbits passing through this compact region can be described using symbolic dynamics. In this case it is shown that a single periodic orbit bifurcates into three periodic orbits whose stable and unstable manifolds intersect transversely. © 1988 Academic Press, Inc.

## 1. INTRODUCTION

In Hamiltonian systems, the existence of transverse intersections between the stable and unstable manifolds of distinct periodic orbits gives rise to regions in which solutions to the system exhibit complex or pathological behavior. For Hamiltonian systems arising from a Hamiltonian function,  $H: \mathbb{R}^4 \rightarrow \mathbb{R}$ , Rod [7], and Churchill and Rod [2, 3, 4] established methods for showing the existence of such transverse intersections. Essential to their methods is the fact that along solutions of a Hamiltonian system the Hamiltonian function (i.e., the total energy) is constant. Thus, by considering a fixed energy level, they need only consider a 3-dimensional system. In the examples considered in [3, 4, 7], the analysis is almost exclusively limited to the energy levels at which the complex behavior is exhibited.

In this paper, we treat the energy level as a bifurcation parameter, and

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present an example in which a single periodic orbit bifurcates into three periodic orbits whose stable and unstable sets intersect transversely. Our example differs from theirs in two other ways. First, we do not assume the existence of any global symmetries (compare with [3, 4, 7]). For the sake of clarity, we introduce a local symmetry at the origin. However, this symmetry is not necessary and we sketch how the results can be established for the more general non-symmetric case. Second, the Hamiltonian functions considered in [3, 4, 7] and here are all of the form  $H(x, y) = \frac{1}{2}\langle y, y \rangle + V(x)$  where  $x, y \in \mathbb{R}^2$ . In the former papers, the origin is either a local minimum of  $V$  or a degenerate saddle point (i.e., there exist directions in which  $V$  increases and other directions in which  $V$  decreases). In our example, the origin is taken to be a non-degenerate local maxima of  $V$ , i.e.,

$$D^2V(0) = \begin{bmatrix} -a_1^2 & 0 \\ 0 & -a_2^2 \end{bmatrix}.$$

As will be seen, the set of pathological bounded orbits lies arbitrarily close to the origin.

The energy level,  $H = 0$ , is the value at which the bifurcation takes place. It is assumed that on this energy level two orbits, homoclinic to the origin, appear. Furthermore, as the energy,  $H$ , is decreased, these homoclinic orbits become disjoint periodic orbits. Section 3 will show that if the homoclinic orbits approach the origin at a certain angle, then there exists another distinct periodic orbit. These three periodic orbits are the "basic" periodic orbits, whose stable and unstable sets are shown to intersect transversely. If we take the "angle" between the homoclinic orbits to be a bifurcation parameter (we are now letting the Hamiltonian function vary), then we have another bifurcation occurring. In this case, two orbits homoclinic to the origin, with their "nested" periodic orbits, bifurcate into a periodic orbit and a region exhibiting the above-mentioned pathological behavior.

Because we assume that the origin is a non-degenerate critical point of  $V$  (and hence  $H$ ), the solutions to Hamilton's equations

$$\begin{aligned} \dot{x}_i &= y_i \\ \dot{y}_i &= -D_i V(x) \end{aligned} \tag{1.1}$$

can be approximated near the origin by solutions to the linear problem defined by  $H_L(x, y) = \frac{1}{2}\langle y, y \rangle - \frac{1}{2}(a_1^2 x_1^2 + a_2^2 x_2^2)$ . This is done in Section 2. In fact, in order to present the concepts clearly, we only consider the linear problem in Section 2.1. To understand how the solutions behave near the origin we restrict our attention to the set  $\{z \mid H(z) = 0\}$  and replace the origin by a torus. This new space is called the critical manifold. On this critical manifold we define a new flow, compatible with that defined by

(1.1), and use this new flow to analyze the behavior of solutions passing near the origin.

Whereas the results of Section 2 are local in nature, Section 3 begins the analysis of the global structure of the solutions. As such we need to introduce the global non-linearities of  $V$ . These are given as a series of assumptions concerning the qualitative behavior of the flow generated by the Hamiltonian system, rather than explicit restrictions on the potential functions. There are reasons for choosing this indirect approach. First, the hypotheses of the theorems are qualitative in nature. Unfortunately, the analysis required to check that a particular potential function satisfies such hypotheses is often long, sometimes difficult, and usually ad hoc. For a discussion on the types of functions which give rise to this qualitative behavior, or on the types of arguments which can be employed to demonstrate such behavior, the reader is referred to [1–4, 7]. Second, in order to obtain pathologies in the manner described in this paper, one needs rather mild assumptions. In fact, most of the assumptions we make can be changed without significantly altering the results. We choose the conditions on  $V$  so as to emphasize the underlying causes of the results rather than to obtain the most general or most easily applicable results.

Also, in Section 3, we restrict our attention, for the most part, to

$$V(x_1, x_2) = -\frac{a^2}{2}(x_1^2 + x_2^2) + V_0(x),$$

where  $V_0(x)$  is  $O(\|x\|^2)$  at the origin. We use the results of Section 2 to prove the existence of an isolated periodic orbit which persists for all energy levels near the bifurcation point  $H=0$ . Changing to the case where the angle between the homoclinic points is used as a bifurcation parameter, we prove (Theorem 3.23) that a periodic orbit bifurcates out of two homoclinic orbits when the angle is  $90^\circ$ . Finally, we comment briefly on how similar results could be obtained for the case  $V(x_1, x_2) = -\frac{1}{2}(a_1^2 x_1^2 + a_2^2 x_2^2) + V_0(x)$ .

The results of Sections 4 and 5 depend heavily on the work of Rod [7] and Churchill and Rod [2]. Section 4 shows that the hypotheses of their theorems are satisfied. Unfortunately, developing the language in which to state the hypotheses is a lengthy process. Thus, rather than repeat a substantial portion of their papers [2, 3, 7], it is assumed that the reader is familiar with their work, and hence, only the results which differ substantially from theirs are proved.

In Section 5, using symbolic dynamics, we classify the set of orbits which intersect a compact region defined by the “basic” periodic and homoclinic orbits near the origin. Our presentation of these results is very curt and the reader is referred to [2, 6, 7] for a more complete interpretation of results of this type.

## 2. THE CRITICAL MANIFOLD

### 2.1. The Linear System

For the sake of clarity of exposition, we begin by considering a linear Hamiltonian system. For our purposes, the most general potential function we can choose is of the form,  $-\frac{1}{2}(a_1^2 x_1^2 + a_2^2 x_2^2)$ . However, since we are only concerned with the qualitative behavior, we can scale out one of the coefficients to get

$$V(x) = V(x_1, x_2) = -\frac{1}{2}(x_1^2 + a^2 x_2^2), \quad a \geq 1. \quad (2.1)$$

This gives rise to the Hamiltonian function,  $H: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$  where

$$H(x, y) = \frac{1}{2}\langle y, y \rangle + V(x). \quad (2.2)$$

If we let  $z = (x, y) \in \mathbb{R}^2 \times \mathbb{R}^2$ , then Hamilton's equations applied to (2.2) give the linear system of differential equations,

$$\dot{z} = Ax, \quad \cdot = d/dt, \quad (2.3)$$

where

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & a^2 & 0 & 0 \end{bmatrix}.$$

By the chain rule we have that  $H$  is constant on solutions of (2.3). Let  $M = \{z \in \mathbb{R}^4 \mid H(z) = 0\}$ . One easily checks that the origin  $O$ , is the only critical point of (2.3) and that  $O \in M$ . In order to understand the behavior of solutions of (2.3) which lie in  $M$  and pass near  $O$ , we will replace the critical point by a critical manifold and define a flow on our new space which corresponds to the old flow on this manifold. This section details the construction of the critical manifold for (2.3). As will be seen in the next section, this construction carries over to the class of non-linear Hamiltonian systems which interest us.

Since (2.3) is a linear system it is possible to rewrite the differential equations in polar form. Let  $S^3(r) = \{z \in \mathbb{R}^4 \mid \|z\| = r\}$  and let  $\zeta \in S^3(1)$ . Given  $z \in \mathbb{R}^4 \setminus \{O\}$  there exists a unique  $r > 0$  and a unique  $\zeta$  such that  $z = r\zeta$ . The polar form differential equations are given in the following lemma.

LEMMA 2.1. (a)  $d/dt \langle \zeta, \zeta \rangle = 2 \langle \dot{\zeta}, \zeta \rangle = 0$ .

$$(b) \quad \dot{r} = r \langle A\zeta, \zeta \rangle. \quad (2.4)$$

$$(c) \quad \dot{\zeta} = A\zeta - \langle A\zeta, \zeta \rangle \zeta. \quad (2.5)$$

*Proof.* (a) This follows from  $\langle \zeta, \zeta \rangle = 1$ .

(b) Differentiating  $z = r\zeta$  gives  $rA\zeta = \dot{r}\zeta + r\dot{\zeta}$ . Taking the inner product with  $\zeta$  and using (a) gives (b).

(c) Substitute (b) into the above expression for  $\dot{z}$ . ■

Understanding the flow given by (2.4) and (2.5) turns out to be of great importance. Of particular interest is the fact that (2.5) is independent of  $r$ . This implies that  $S^3(r)$  is an invariant set for (2.5), for all  $r \geq 0$ , which is not surprising since (2.5) is nothing more than the radial projection of the flow of (2.2) onto the unit sphere centered at the origin. Since  $S^3(r)$  is an invariant set of (2.5) it makes sense to ask how the solutions restricted to  $S^3(r) \cap M$  behave. Let  $P_1: M \rightarrow \mathbb{R}^2$  be given by  $P_1(x, y) = x$ . Let  $T(r) = S^3(r) \cap M$  and define  $K(r) = \{x \mid 2x_1^2 + (1 + a^2)x_2^2 = r\} = P_1(T(r))$ .

PROPOSITION 2.2.  $T(r)$  is homeomorphic to a torus, i.e.,  $S^1 \times S^1$ .

*Proof.*  $K(r)$  defines an ellipse since  $r > 0$ , and hence is homeomorphic to a circle, i.e.,  $S^1$ . Given  $x \in K(r)$ , one can check that

$$P_1^{-1}(x) = \{(x, y) \mid \langle y, y \rangle = \frac{1}{2}(r + (a^2 - 1)x_2^2)\}.$$

This implies that for all  $x \in K(r)$ ,  $P_1^{-1}(x)$  is homeomorphic to  $S^1$ . We can think of  $T(r)$  as a fiber bundle with base  $K(r) \sim S^1$  and fiber  $P_1^{-1}(x) \sim S^1$ . Since  $y$  corresponds to the velocity vector it is clear that  $T(r)$  is orientable. Thus  $T(r)$  is a torus. ■

The eigenvalues of  $A$  are  $\pm 1$  and  $\pm a$ . The corresponding eigenvector spaces are generated by  $(x_1, 0, x_1, 0)$ ,  $(x_1, 0, -x_1, 0)$ ,  $(0, x_2, 0, ax_2)$ , and  $(0, x_2, 0, -ax_2)$ . The stable manifold to the origin,  $W^s$ , is spanned by  $(x_1, 0, -x_1, 0)$  and  $(0, x_2, 0, -ax_2)$ . The unstable manifold,  $W^u$ , is spanned by  $(x_1, 0, x_1, 0)$  and  $(0, x_2, 0, ax_2)$ . Thus both  $W^u$  and  $W^s$  are two dimensional manifolds contained in  $M$ . Let

$$S^u(r) = W^u \cap S^3(r) \quad \text{and} \quad S^s(r) = W^s \cap S^3(r).$$

The following is obvious.

LEMMA 2.3. (a)  $S^u(r) \subset T(r)$  and  $S^s(r) \subset T(r)$  for all  $r \geq 0$ .

(b)  $S^u(r)$  and  $S^s(r)$  are homeomorphic to  $S^1$ .

Since (2.5) is independent of  $r$ , the description of the flow on  $T(r)$  will be the same regardless of what value of  $r > 0$  is chosen. Hence, without loss of generality, one can, for purposes of simplifying the calculations, let  $r = 1$ . To simplify the notation let  $T = T(1)$ ,  $S^3 = S^3(1)$ ,  $S^u = S^u(1)$ , and  $S^s = S^s(1)$ .

LEMMA 2.4. *Solutions to (2.5) have the following properties:*

- (a) *If  $\zeta$  is an eigenvector of  $A$  then  $\dot{\zeta} = 0$ .*
- (b) *If  $\zeta \in T$  and  $\zeta \notin S^u \cup S^s$ , then  $\dot{\zeta} \neq 0$ .*
- (c) *Solutions on  $T \setminus \{S^u \cup S^s\}$  are heteroclinic orbits from fixed points of  $S^s$  to fixed points of  $S^u$ .*

*If  $a > 1$ , then the fixed points on  $T$  are the eigenvectors of  $A$  which lie on  $T$ . Furthermore, the flow on  $S^s$  consists of the four critical points  $\pm 2^{-1/2}(1, 0, -1, 0)$  and  $\pm(1+a^2)^{-1/2}(0, 1, 0, -a)$  plus heteroclinic orbits from  $(1+a^2)^{-1/2}(0, 1, 0, -a)$  to  $\pm 2^{-1/2}(1, 0, -1, 0)$  and from  $(1+a^2)^{-1/2}(0, -1, 0, a)$  to  $\pm 2^{-1/2}(1, 0, -1, 0)$ . The flow on  $S^u$  consists of the critical points  $\pm 2^{-1/2}(1, 0, 1, 0)$  and  $\pm(1+a^2)^{-1/2}(0, 1, 0, a)$  plus heteroclinic orbits from  $2^{-1/2}(1, 0, 1, 0)$  to  $\pm(1+a^2)^{-1/2}(0, 1, 0, a)$  and from  $2^{-1/2}(-1, 0, -1, 0)$  to  $\pm(1+a^2)^{-1/2}(0, 1, 0, a)$ .*

*If  $a = 1$  then all the elements of  $S^s \cup S^u$  are fixed points.*

*Proof.* All the results are evident if one recalls that (2.5) is the projection of the linear flow (2.3) onto the unit sphere. ■

Since (2.5) is derived from a linear system, one might hope to be able to find a simple exact description of the heteroclinic orbits on  $T$ . If one assumes that  $a = 1$  then this is the case. For  $a > 1$  we shall not attempt to do so except for a few special orbits.

Notation 2.5. From now on  $\xi$  will denote a critical point in  $S^s$  and  $\eta$  a critical point in  $S^u$ . For  $a > 1$ , the possible values of  $\xi$  and  $\eta$  are given in Lemma 2.4. If  $a = 1$  then  $\xi = (\xi_1, \xi_2, -\xi_1, -\xi_2)$  and  $\eta = (\eta_1, \eta_2, \eta_1, \eta_2)$ .

DEFINITION 2.6. For fixed  $\xi$  and  $\eta$  define a path in  $S^3$  by

$$w(\xi, \eta) = w: [0, 1] \rightarrow S^3$$

where

$$w(\xi, \eta; c) = \frac{(1-c)\xi + c\eta}{\|(1-c)\xi + c\eta\|}.$$

For fixed  $\xi$  and  $\eta$ , define  $\Gamma(c) = \|(1-c)\xi + c\eta\|^{-1}$  and let  $W(c) = \Gamma^{-1}(c)w(c)$ . Finally, let  $w(c) = (w_1(c), w_2(c), w_3(c), w_4(c))$ .

**PROPOSITION 2.7.**  $w(\xi, \eta): [0, 1] \rightarrow T$  if and only if  $\eta = \pm(\xi_2, -\xi_1, -(1/a)\xi_4, a\xi_3)/\|(\xi_2, -\xi_1, -(1/a)\xi_4, a\xi_3)\|$ .

*Proof.* One needs to find the conditions on  $\xi$  and  $\eta$  such that for all  $c \in [0, 1]$ ,  $H(w(c)) = 0$ . By (2.2) this is the same as requiring that

$$w_3^2(c) + w_4^2(c) - w_1^2(c) - a^2 w_2^2(c) = 0. \quad (2.6)$$

At this point there is a multitude of cases which need to be checked. If  $a > 1$  then the results follow by simple substitution. We shall demonstrate the case in which  $a = 1$ .

Substitution of Notation 2.5 and Definition 2.6 into Eq. (2.6) plus some simple calculations yield

$$\xi_1 \eta_1 = -\xi_2 \eta_2 \quad (2.7)$$

or

$$\xi_1^2 \eta_1^2 = \xi_2^2 \eta_2^2. \quad (2.8)$$

Since  $\xi, \eta \in S^3$ , one has that

$$2(\xi_1^2 + \xi_2^2) = 1 = 2(\eta_1^2 + \eta_2^2). \quad (2.9)$$

Using (2.9) to solve for  $\xi_1^2$  and  $\eta_2^2$  and substituting into (2.8) gives  $\eta_1 = \pm \xi_2$ . A similar argument gives  $\eta_2 = \pm \xi_1$ . The desired result now follows from (2.7) and (2.8). ■

**DEFINITION 2.8.** Given  $\xi$ , define  $\eta_+ = (\xi_2, -\xi_1, -(1/a)\xi_4, a\xi_3)/\|(\xi_2, -\xi_1, -(1/a)\xi_4, a\xi_3)\|$  and  $\eta_- = (-\xi_2, \xi_1, (1/a)\xi_4, -a\xi_3)/\|(-\xi_2, \xi_1, (1/a)\xi_4, -a\xi_3)\|$ .

**PROPOSITION 2.9.** The curves  $w(\xi, \eta_+; c)$  and  $w(\xi, \eta_-; c)$ , for  $c \in (0, 1)$ , represent heteroclinic orbits on  $T \setminus \{S^s \cup S^u\}$ . If  $a = 1$  these are all the heteroclinic orbits described in Lemma 2.4.

*Proof.* Again we are faced with a multitude of cases. We shall give the proof for  $a = 1$  and  $\eta_+$ . The other cases follow in a similar manner. Let  $V$  be the vector space spanned by  $\xi$  and  $\eta_+$ .  $w(c)$  is a curve which lies in  $V \cap S^3$ . If for any fixed  $c$ ,  $\zeta = w(c)$  and  $\dot{\zeta} \in V$ , then  $w(c)$  will represent a heteroclinic orbit. Equation (2.5) gives

$$\begin{aligned} \dot{\zeta} &= A w(c) - \langle A w(c), w(c) \rangle w(c) \\ &= \Gamma A W(c) - \Gamma^3 \langle A W(c), W(c) \rangle W(c). \end{aligned}$$

What needs to be shown is that there exist real numbers  $d$  and  $e$  such that  $d\check{\xi} + e\eta_+ = \check{\xi}$ . Simple but tedious calculations give that

$$AW = (c\check{\xi}_2 - (1-c)\check{\xi}_1, -(1-c)\check{\xi}_2 - c\check{\xi}_1, (1-c)\check{\xi}_1 + c\check{\xi}_2, \\ (1-c)\check{\xi}_2 - c\check{\xi}_1),$$

$$\langle AW, W \rangle = 2c - 1,$$

$$d = -\Gamma(1-c)[1 + (2c-1)\Gamma^2],$$

and

$$e = \Gamma c[1 - (2c-1)\Gamma^2].$$

The details of checking that  $d\check{\xi} + e\eta_+ = \check{\xi}$  is satisfied is left to the reader. ■

Consider (2.4) restricted to  $T$ . Notice that  $z = r\check{\xi} \in S^+$  implies  $\dot{r} < 0$  and  $z \in S^-$  implies  $\dot{r} > 0$ . We want to describe the set of points on  $S^3(r)$  at which  $\dot{r} = 0$ . For  $r > 0$  this means solving  $\langle A\check{\xi}, \check{\xi} \rangle = 0$ . One easily checks that if  $z$  lies in the vector space spanned by

$$\{(1, \sqrt{2}(a(1+a^2))^{-1/2}, 1, -\sqrt{2}a(1+a^2)^{-1/2}), (1, \sqrt{2}(a(1+a^2))^{-1/2}, \\ -1, \sqrt{2}a(1+a^2)^{-1/2})\}$$

or, by

$$\{(-1, \sqrt{2}(a(1+a^2))^{-1/2}, 1, -\sqrt{2}a(1+a^2)^{-1/2}), (-1, \sqrt{2}(a(1+a^2))^{-1/2}, \\ 1, \sqrt{2}a(1+a^2)^{-1/2})\}$$

then  $\langle Az, z \rangle = 0$ ; i.e., if  $z = r\check{\xi}$  then  $\dot{r} = 0$ . On the other hand, since  $\nabla_z \langle Az, z \rangle \neq 0$  for all  $z \neq 0$ , these planes are the only points for which  $\dot{r} = 0$ . These two planes intersect  $T(r)$  in two disjoint circles. Figure 1 describes the flow of (2.5) restricted to  $T(r)$  for  $a > 1$ . If  $a = 1$  then  $\dot{r} = 0$  at  $w(\check{\xi}, \eta; c)$  if and only if  $c = \frac{1}{2}$ . If  $c \in [0, \frac{1}{2})$ , then  $r < 0$  and if  $c \in (\frac{1}{2}, 1]$  then  $r > 0$ .

So far, the results of this section have been independent of  $r$ , the only restriction being that  $r > 0$ . Now consider the case  $r = 0$ . In  $M$ ,  $r = 0$  corresponds to the origin which, according to (2.3), is a rest point. While (2.5) is still applicable when  $r = 0$ , it is of limited use when applied to a single point. Thus, to fully exploit (2.5) it is necessary to construct a critical manifold,  $CM$ , to replace  $M$ . In particular the origin in  $M$ ,  $O$  will be replaced by a torus,  $CT$ , on which (2.5) is defined in a non-trivial manner. The details are what follows.

Let  $A \amalg B$  denote the disjoint union of two sets  $A$  and  $B$ .



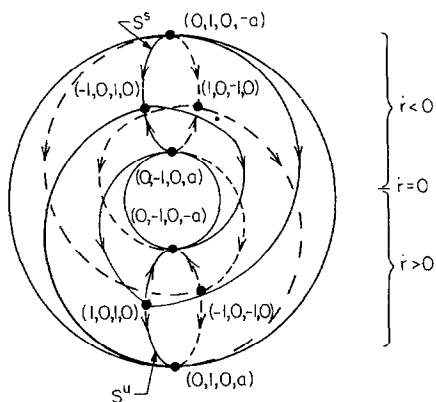


FIGURE 1

DEFINITION 2.10. Let  $X = \mathbb{R}^4 \setminus \{O\} \amalg S^3$ . Define  $h: X \rightarrow \{v \in \mathbb{R}^4 \mid \|v\| \geq 1\}$  by

$$h(z) = h(r\zeta) = \begin{cases} (r+1)\zeta & \text{if } z \in \mathbb{R}^4 \setminus \{O\} \\ \zeta & \text{if } z \in S^3. \end{cases}$$

The topology of  $X$  is such that  $h$  is a homeomorphism.

The following system of coordinates will be used to describe the elements of  $X$ . If  $z \in X$  then  $z \in \mathbb{R}^4 \setminus \{O\}$  or  $z \in S^3$ . In the first case, one writes  $z = r\zeta = (r, \zeta)$ . In the latter case, one writes  $z = (0, \zeta)$ . Using this notation, one can check that (2.4) and (2.5) are well defined on  $X$ . Furthermore, (2.4) and (2.5) give rise to a continuous flow on  $X$ .

Since  $M \setminus \{O\} \subset \mathbb{R}^4 \setminus \{O\}$  there is an obvious embedding of  $M \setminus \{O\}$  into  $X$  given by  $z = r\zeta \mapsto (r, \zeta)$ . In addition, under this embedding  $M \setminus \{O\}$  is not closed in  $X$ . Define  $CM$  to be the closure of  $M \setminus \{O\}$  in  $X$ . Let

$$CT = CM \setminus (M \setminus \{O\}).$$

PROPOSITION 2.11.  $CT$  is homeomorphic to a torus.

*Proof.* Notice that  $CT \subset S^3 \subset X$ , hence  $z \in CT$  implies that  $z = (0, \zeta)$ . Let  $\zeta \in T$  then  $(r, \zeta) \in M \setminus \{O\}$  for all  $r > 0$ . Thus

$$\lim_{r \rightarrow 0} (r, \zeta) = (0, \zeta) \in CT.$$

If  $\zeta \in S^3 \setminus T$  then  $(0, \zeta)$  is not a limit point of  $M \setminus \{O\}$  in  $X$ , thus  $(0, \zeta) \notin CT$ . Therefore  $CT$  and  $T$  are homeomorphic which by Proposition 2.2 implies that  $CT$  is homeomorphic to a torus. ■

The flow on  $CM$  is determined by (2.4) and (2.5). Notice that for  $(r, \zeta)$ ,  $r > 0$ , one has the same flow as that determined by (2.3). However, on  $CT$  the flow arises from (2.4) and (2.5) for  $r = 0$ . Since elements of  $CT$  are of the form  $(0, \zeta)$ ,  $CT$  is an invariant set of the flow. The flow is continuous on  $CM$  since (2.5) is independent of  $r$  and (2.4) is continuous in  $r$  for all  $r \geq 0$ . This flow, which is a mapping  $CM \times \mathbb{R} \rightarrow CM$ , will be denoted by

$$((r, \zeta), t) \mapsto (r, \zeta) \cdot t = (r \cdot t, \zeta \cdot t),$$

where  $r \cdot t$  is determined by (2.4) and  $(\zeta \cdot t)$  is determined by (2.5).

DEFINITION 2.12. For fixed  $r_0 > 0$  define

$$B(r_0) = \{(r, \zeta) \in CM \mid r \leq r_0\}$$

$$B^s(r_0) = \{(r_0, \zeta) \in CM \mid \dot{r} \leq 0\}$$

$$B^u(r_0) = \{(r_0, \zeta) \in CM \mid \dot{r} \geq 0\}.$$

REMARKS 2.13. (a)  $T(r) = B^s(r) \cup B^u(r)$ .

(b) It follows from Proposition 2.2, Lemmas 2.3, and 2.4, and (2.5) that  $B^s(r)$  and  $B^u(r)$  are homeomorphic to annuli.

(c)  $B^s(r) \cap B^u(r)$  is homeomorphic to two disjoint circles (see comments following Proposition 2.9) which will be denoted by  $C_1(r)$  and  $C_2(r)$ .

(d) In the language of Conley [5],  $B(r)$ ,  $r > 0$  is an isolating neighborhood for the maximal invariant set  $CT$ . The exit set and entrance set for  $B(r)$  are given by  $B^u(r)$  and  $B^s(r)$ , respectively. Moreover,  $C_1(r)$  and  $C_2(r)$  are "tangency sets."

PROPOSITION 2.14. Given  $(r, \zeta) \in B^s(r) \setminus S^s(r)$  there exists a unique  $t^* = t^*(r, \zeta) \geq 0$  such that  $(r, \zeta) \cdot t^* \in B^u(r) \setminus S^u(r)$  and  $(r, \zeta) \cdot [0, t^*] \subset B(r)$ . Define  $\varphi: B^s(r) \setminus S^s(r) \rightarrow B^u(r) \setminus S^u(r)$  by  $\varphi(r, \zeta) = (r, \zeta) \cdot t^*(r, \zeta)$ . Then  $\varphi$  is a homeomorphism.

Proof. Because the flow defined on  $CM$  corresponds to the flow of the linear equations (2.3) for all elements except those on  $CT$ , it is clear that if  $(r, \zeta) \cdot \mathbb{R} \subset B(r)$  then  $(r, \zeta) \in T(0)$ . Also, from the linearity of (2.3), one has that if  $(r, \zeta) \cdot t \in B(r)$  for all  $t \geq 0$  then  $(r, \zeta) \in S^s(r)$ . Likewise, if  $(r, \zeta) \cdot t \in B(r)$  for all  $t < 0$  then  $(r, \zeta) \in S^u(r)$ . Thus, one has the existence of  $t^*$  if  $(r, \zeta) \in B^s(r) \setminus S^s(r)$ , and one has that

$$\varphi(B^s(r) \setminus S^s(r)) = B^u(r) \setminus S^u(r).$$

That  $\varphi$  is a homeomorphism follows from the continuous dependence on initial conditions of solutions of ordinary differential equations. ■

COROLLARY 2.15.  $\varphi$  is the identity on  $C_1(r) \cup C_2(r) = B^s(r) \cap B^u(r)$ .

For the moment we restrict our attention to the case  $a = 1$ . In this case, a typical element of  $S^s(r)$  must be of the form  $(r, \xi)$ . Furthermore,  $\lim_{t \rightarrow \infty} (r, \xi) \cdot t = (0, \xi) \in S^s(0) \subset CT$ . By Proposition 2.9 and Definition 2.8,  $w(\xi, \eta_{\pm}; c)$  describes the two heteroclinic orbits on  $CT$  connecting  $(0, \xi) \in S^s(0)$  to  $(0, \eta_{\pm}) \in S^u(0)$ . In addition,  $\lim_{t \rightarrow -\infty} (r, \eta_{\pm}) = (0, \eta_{\pm})$ . This allows us to define the following paths in  $CM$ .

DEFINITION 2.16. Given any  $(r, \xi) \in S^s$  let  $\gamma_{\pm}(\xi): [0, 3] \rightarrow B(r)$  denote the two paths defined by

$$\begin{aligned} \gamma_{\pm}(\xi)(s) &= (r, \xi) \cdot (s/1 - s) & \text{for } s \in [0, 1) \\ \gamma_{\pm}(\xi)(1) &= (0, \xi) \\ \gamma_{\pm}(\xi)(s) &= (0, w(\xi, \eta_{\pm}; s-1)) & \text{for } s \in (1, 2) \\ \gamma_{\pm}(\xi)(2) &= (0, \eta_{\pm}) \\ \gamma_{\pm}(\xi)(s) &= (r, \eta_{\pm}) \cdot (3-s)/(2-s) & \text{for } s \in (2, 3]. \end{aligned}$$

DEFINITION 2.17. Let  $d$  be a metric on  $CM$ . Let  $\gamma: [0, 3] \rightarrow CM$ . Let  $(r, \zeta) \in CM$ . One says that the orbit of  $(r, \zeta)$  lies in an  $\varepsilon$ -tube about  $\gamma$  over the interval  $[t_0, t_1]$  if for all  $t \in [t_0, t_1]$ :

$$\inf_{s \in [0, 3]} \{d(\gamma(s), (r, \zeta) \cdot t)\} < \varepsilon.$$

Recall that the purpose of the critical manifold is to describe the behavior of solutions of (2.3) near the origin. This will be done by demonstrating how  $\varphi$  acts on  $(r, \eta) \in B^s(r) \setminus S^s(r)$  if  $(r, \eta)$  lies close to  $S^s(r)$ . By Proposition 2.14 there exists a unique maximal  $t^* = t^*(r, \zeta)$  such that  $(r, \zeta) \cdot [0, t^*] \subset B$ . Let  $(r, \xi) \in S^s(r)$ . Recall the construction of  $\gamma_{\pm}(\xi)$ . Each "piece" of  $\gamma_{\pm}$  is made up of a solution to (2.4) and (2.5). Thus, by continuity one can choose  $(r, \zeta) \in B^s \setminus S^s$  such that the orbit of  $(r, \zeta)$  lies in an  $\varepsilon$ -tube of either  $\gamma_+(\xi)$  or  $\gamma_-(\xi)$  over the interval  $[0, t^*]$ .

Remark 2.18. A technically incorrect but intuitively illuminating restatement of what we have just shown is as follows. If one considers the path of a solution to (2.3) in  $x$ -space, then those orbits which lie on the stable manifold to the origin, leave the origin on the unstable manifold in a direction perpendicular to the direction of entry. In application, this means that orbits on the surface  $M$  which pass close to the origin change direction by slightly less than  $90^\circ$  (see Fig. 2).

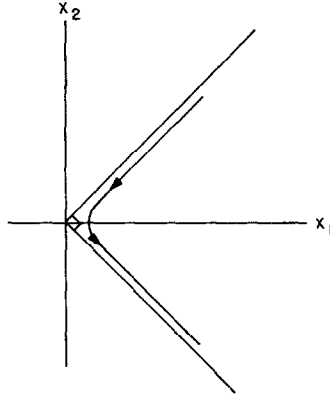


FIGURE 2

We now consider briefly the case  $a > 1$ . Let

$$\begin{aligned}\pm \xi^1 &= \pm \left( 0, \frac{1}{\sqrt{1+a^2}}, 0, \frac{-a}{\sqrt{1+a^2}} \right), \\ \pm \xi^2 &= \pm \left( \frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}, 0 \right), \\ \eta_{\pm}^1 &= \pm \left( 0, \frac{1}{\sqrt{1+a^2}}, 0, \frac{a}{\sqrt{1+a^2}} \right)\end{aligned}$$

and  $\eta_{\pm}^2 = \pm (1/\sqrt{2}, 0, 1/\sqrt{2}, 0)$ . Let  $(r, \zeta) \in S^3(r)$ . If  $\zeta \neq \pm \xi^1$  then  $\lim_{t \rightarrow \infty} (r, \eta) \cdot t = (0, \pm \xi^2)$ . Assume  $\lim_{t \rightarrow \infty} (r, \zeta) \cdot t = (0, \xi^2)$ . Then we can define  $\gamma_{\pm}(\zeta): [0, 3] \rightarrow B(r)$  as follows:

$$\begin{aligned}\gamma_{\pm}(\zeta)(s) &= (r, \zeta) \cdot (s/1 - s) & \text{for } s \in [0, 1) \\ \gamma_{\pm}(\zeta)(1) &= (0, \xi^2) \\ \gamma_{\pm}(\zeta)(s) &= (0, w(\xi^2, \eta_{\pm}^1; s-1)) & \text{for } s \in (1, 2) \\ \gamma_{\pm}(\zeta)(2) &= (0, \eta_{\pm}^1) \\ \gamma_{\pm}(\zeta)(s) &= (r, \eta_{\pm}^1) \cdot (3-s)/(2-s) & \text{for } s \in (2, 3].\end{aligned}$$

If  $\lim_{t \rightarrow \infty} (r, \zeta) \cdot t = (0, -\xi^2)$  then there is a corresponding definition for  $\gamma_{\pm}(\zeta)$ . The question of how to define  $\gamma_{\pm}(\pm \xi^1)$  is more delicate. Because the system is linear, we have an exact solution for  $(2, 3)$ , namely,  $z(t) = e^{At} z_0$ . Using this, we conclude that  $\gamma_{\pm}$  should satisfy

$$\gamma_{\pm}(\xi^1)(2) = (0, \eta_{\pm}^2) = \gamma_{\mp}(-\xi^1)(2).$$

(This involves checking that  $(r, \zeta) \in CM$  close to  $(r, \xi^1)$  implies that  $(r, \zeta) \cdot t^*(r, \zeta)$  is not close to  $(r, \eta_\pm^1)$ ). Again, by continuity  $\gamma_\pm$  can be used to describe the path of  $(r, \eta) \in B^s(r) \setminus S^s(r)$  if  $(r, \eta)$  lies sufficiently close to  $S^s(r)$ .

## 2.2. The Nonlinear System

The results of Section 2.1 are easily extendable to the class of non-linear Hamiltonian functions,  $H \in C^2(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R})$ , given by

$$H(x, y) = \frac{1}{2} \langle y, y \rangle + V(x), \quad (2.10)$$

where

$$V(x) = -\frac{1}{2}(a_1^2 x_1^2 + a_2^2 x_2^2) + V_0(x) \quad (2.11)$$

and  $V_0(x) = o(\|x\|^2)$  at the origin. As before, we note that the qualitative picture near the origin will not change if we set  $a_1 = 1$ . The differential equations of interest are given by Hamilton's equation, i.e.,

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\nabla V(x). \end{aligned} \quad (2.12)$$

As in the case of (2.3), the origin,  $O$ , is a fixed point for (2.12) and  $H$  (i.e., (2.10)) is constant along solutions. Thus  $M = \{z \mid H(z) = 0\}$  is invariant under (2.12). As before we can embed  $M \setminus \{O\}$  into  $X$  and define  $CM$  to be the closure of  $M \setminus \{O\}$  in  $X$ . Define  $CT = CM \setminus (M \setminus \{O\})$ . The question is whether the flow on  $CT$  determined by (2.12) is the same as that of (2.3). The following theorem answers it in the affirmative.

**THEOREM 2.19.** (Hartman-Grobman; see [6, p.13]). *Let  $z = f(z)$ ,  $z \in \mathbb{R}^n$  and  $f(\bar{z}) = 0$ . If  $Df(\bar{z})$  has no zero or purely imaginary eigenvalues then there is a homeomorphism  $h$  defined on some neighborhood  $U$  of  $\bar{z}$  in  $\mathbb{R}^n$  locally taking orbits of the nonlinear flow  $\varphi_t$ , corresponding to  $\dot{z} = f(z)$ , to those of the linear flow  $e^{tDf(\bar{z})}$  corresponding to  $\dot{z} = Df(\bar{z})z$ . The homeomorphism preserves the sense of orbits and can also be chosen to preserve parametrization by time.*

## 3. PERIODIC ORBITS

This section contains theorems concerning the existence of periodic orbits arising from the Hamiltonian systems of the form (2.12). It is worth emphasizing that these periodic orbits do not occur because of a global

symmetry but rather because of the local behavior described in Section 2. For the sake of clarity in our exposition we shall assume

$$V(x) = -\frac{1}{2}(x_1^2 + x_2^2) + V_0(x). \quad (3.1)$$

This is a strong assumption as it introduces local symmetry. However, it is not crucial to the types of arguments used in the proofs that follow.

Let  $E(h) = \{x \mid V(x) = h\}$ , then  $E$  is called an *equipotential set* of  $V$ . Let  $h^- < 0 < h^+$ .  $h^-$  and  $h^+$  will be lower and upper bounds for  $h$  and need to be chosen sufficiently small for the following results to hold. We do not attempt to estimate what these values should be.

ASSUMPTION 1. (a) If  $h \in (h^-, 0)$  then  $E(h) = EO(h) \cup ET(h) \cup EB(h)$  where  $EO(h)$ ,  $ET(h)$  and  $EB(h)$  are disconnected curves in  $\mathbb{R}^2$ . (See Fig. 3.) If  $x = (x_1, x_2) \in ET(h)$  then  $x_2 > 0$ , and if  $x \in EB(h)$  then  $x_2 < 0$ . Furthermore  $ET(h)$  and  $EB(h)$  bound the regions  $\{x \mid V(x) > h\}$  away from the  $x_1$ -axis.  $EO(h)$  is the boundary of a region  $\{x \mid V(x) > h\}$  which contains the origin.

(b) If  $h = 0$  then  $E(h)$  is as in (a). However,  $EO(h)$  is the origin.

(c) If  $h \in (0, h^+)$  then  $E(h)$  is as in (a). However,  $EO(h)$  is the empty set.

To simplify the notation, if  $h = 0$  let  $E = E(0)$ ,  $ET = ET(0)$ , and  $EB = EB(0)$ . Let  $u: [0, 1] \rightarrow ET$ ,  $v: [0, 1] \rightarrow EB$  be parametrizations of portions of  $ET$  and  $EB$ , respectively (see Fig. 4).

As before, the origin in  $\mathbb{R}^4$  is a critical point with 2-dimensional stable and 2-dimensional unstable manifolds denoted by  $W^s$  and  $W^u$ , respectively. Let  $P_i(h): \{z \mid H(z) = h\} \rightarrow \mathbb{R}^2$ ,  $i = 1, 2$ , be given by  $P_1(h)(x, y) = x$  and  $P_2(h)(x, y) = y$ . Again, to simplify the notation we write  $P_i = P_i(0)$ . It is easily checked that if  $H(x, y) = h$  and  $x \in E(h)$  then  $y = 0$ . Thus, no confusion should arise if one considers  $E(h) \subset \mathbb{R}^2$  or  $E(h) \subset$

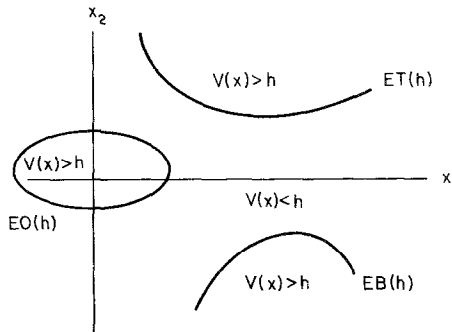


FIGURE 3

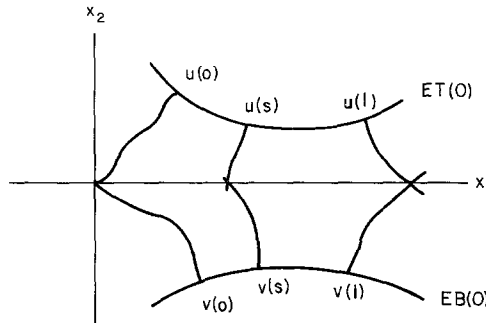


FIGURE 4

$\{(x, y) | H(x, y) = h\}$ . In particular,  $u(s)$  and  $v(s)$  will be used interchangeably to denote elements of  $\mathbb{R}^2$  and elements of  $M$ .

DEFINITION 3.1. Given  $z \in \mathbb{R}^4$  such that  $H(z) = h$ , define

$$\lambda(z) = \inf\{t > 0 | P_1(h)(z \cdot t) \text{ lies on the } x_1\text{-axis}\}.$$

ASSUMPTION 2. (a)  $u(0)$  and  $v(0)$  lie on the stable manifold of the origin.

(b)  $\lambda(u(s))$  and  $\lambda(v(s))$  exist for all  $s \in (0, 1]$ .

(c)  $P_1(u(s) \cdot \lambda(u(s))) = P_1(v(s) \cdot \lambda(v(s)))$  for all  $s \in (0, 1]$  (See Fig. 4).

Let  $P_i^*: X \rightarrow [-1, 1] \times [-1, 1]$  for  $i = 1, 2$  where  $P_1^*(r, \zeta) = (\zeta_1, \zeta_2)$  and  $P_2^*(r, \zeta) = (\zeta_3, \zeta_4)$ . By assumption 2(a),  $u(0)$  and  $v(0)$  are elements of  $W^s$ . Thus  $u(0)$  and  $v(0)$  are elements of  $M$ . By the conventions of Section 2

$$\begin{aligned} \lim_{t \rightarrow \infty} u(0) \cdot t &= (0, \xi(u)) \in CT \\ \lim_{t \rightarrow \infty} v(0) \cdot t &= (0, \xi(v)) \in CT. \end{aligned} \tag{3.2}$$

DEFINITION 3.2. Let  $\Phi$ ,  $0^\circ \leq \Phi \leq 180^\circ$ , be the angle defined by

$$\cos \Phi = \frac{P_1^*(\xi(u))}{\|P_1^*(\xi(u))\|} \cdot \frac{P_1^*(\xi(v))}{\|P_1^*(\xi(v))\|}.$$

ASSUMPTION 3.  $0^\circ < \Phi < 90^\circ$ .

ASSUMPTION 4. If  $(x_1(t), x_2(t)) = P_1(u(0) \cdot t)$  then  $x_1(t) > 0$  and  $x_2(t) > 0$  for all  $t \in [0, \infty)$ . If  $(x_1(t), x_2(t)) = P_1(v(0) \cdot t)$  then  $x_1(t) > 0$  and  $x_2(t) < 0$  for all  $t \in [0, \infty)$ .

DEFINITION 3.3. Let  $\alpha(s) \in [-\pi, 0]$  and  $\beta(s) \in [0, \pi]$  for  $s \in (0, 1]$  be defined by

$$\cos(\alpha(s)) = \frac{\langle P_2(u(s) \cdot \lambda(u(s))), (1, 0) \rangle}{\|P_2(u(s) \cdot \lambda(u(s)))\|}$$

$$\cos(\beta(s)) = \frac{\langle P_2(v(s) \cdot \lambda(v(s))), (1, 0) \rangle}{\|P_2(v(s) \cdot \lambda(v(s)))\|}.$$

Geometrically,  $\alpha(s)$  and  $\beta(s)$  represent the angles through which the orbits originating at  $u(s)$  and  $v(s)$ , respectively, cross the  $x_1$ -axis for the first time.

DEFINITION 3.4.  $\Psi(s) = \beta(s) - \alpha(s)$ .

ASSUMPTION 5.  $\Psi(1) < \pi$ .

In order to emphasize the difference between  $\Phi$ , the angle made by the homoclinic orbits which pass through  $u(0)$  and  $v(0)$ , and  $\Psi$ , the angle made by orbits crossing the  $x_1$ -axis, we shall always measure  $\Phi$  in degrees and  $\Psi$  in radians.

THEOREM 3.5. *Given assumptions 1–5, there exists at least one periodic orbit on the energy surface  $\{z \mid H(z) = 0\}$ .*

The proof of this theorem is straightforward once one deals with the following two technicalities. First, at present  $\Psi$  is only defined for  $s \in (0, 1]$ . One needs to extend the definitions of  $\alpha$  and  $\beta$  in such a way that they are continuous functions on the closed interval  $[0, 1]$ . This in turn will mean that  $\Psi$  is a continuous function on  $[0, 1]$ . Second, one needs to know the value of  $\Psi(0)$ . As will be shown, Assumption 3 forces  $\Psi(0) > \pi$ . Assume these problems have been dealt with.

*Proof.* Since  $\Psi$  is continuous on  $[0, 1]$ ,  $\Psi(0) > \pi$ , and  $\Psi(1) < \pi$ , there exists an  $s^* \in (0, 1)$  such that  $\Psi(s^*) = \pi$ . This in turn implies that the orbit passing through  $u(s^*)$  crosses the  $x_1$ -axis in exactly the opposite direction from the orbit passing through  $v(s^*)$ . Since  $V(u(s^*)) = 0 = V(v(s^*))$ , the velocity at these points is zero. Invoking the reversibility of Hamiltonian systems, one has that the same orbit passes through  $u(s^*)$  and  $v(s^*)$  and hence that this orbit is periodic. ■

The critical manifold can be used to define  $\alpha(0)$  and  $\beta(0)$  in such a way that  $\alpha$  and  $\beta$  are continuous functions in  $[0, 1]$ . For the moment, consider only the function  $\alpha$ . As was mentioned before,  $\alpha(s)$  represents the angle at which the orbit originating at  $u(s)$  crosses the  $x_1$ -axis for the first time. By



(3.2)  $\lim_{t \rightarrow \infty} u(0) \cdot t = (0, \xi(u)) \in CT$ . Thus we have the two curves  $\gamma_{\pm} : [0, 3] \rightarrow CM$  defined in Section 2 such that  $\gamma_{\pm}(0) = u(0)$ ,  $\gamma_{\pm}(1) = (0, \xi(u))$ , and  $\gamma_{\pm}(2) = (0, \eta_{\pm}(u))$ .

**PROPOSITION 3.6.** *For  $s$  sufficiently small, the orbit of  $u(s)$  lies in an  $\varepsilon$ -tube about  $\gamma_+$  over the interval  $[0, t^*]$ , where  $t^*$  is as in Proposition 2.14.*

*Proof.* By continuity of the flow, given any  $\varepsilon > 0$ , there exists  $t^* > 0$ , such that for  $s$  sufficiently small the orbit  $u(s)$  lies in an  $\varepsilon$ -tube about  $\gamma_+$  or  $\gamma_-$  over the interval  $[0, t^*]$ . Assume the latter, i.e.,  $u(s)$  is close to  $\gamma_-$ . For all  $s \in (0, 1]$  one has that  $P_1^*(u(s) \cdot \lambda(u(s))) = (x_1, 0)$  where  $x_1 > 0$ . Hence one must be able to solve  $P_1^*(w(\xi(u), \eta_-(u); c)) = (x_1, 0)$  where  $x_1 > 0$ . We shall show that this is not possible.

Let  $\xi(u) = (p, mp, -p, -mp)$  where  $p > 0$  and  $m > 0$ . That  $\xi(u)$  must be of this form follows from Assumption 4 and Notation 2.5. This in turn implies  $\eta_-(u) = (-mp, p, -mp, p)$ . Thus, in order for  $P_1^*(w(\xi(u), \eta_-(u); c)) = (x_1, 0)$ , it must be the case that  $(1 - c)mp + cp = 0$  thus  $c = m/(m - 1)$ . But  $c \in (0, 1)$  hence  $m < 0$ , a contradiction. ■

A straightforward calculation gives:

**PROPOSITION 3.7.** *If  $\xi(u) = (p, mp, -p, -mp)$  where  $p > 0$  and  $m > 0$  and if  $P_1^*(w(\xi(u), \eta_+(u); c)) = (x_1, 0)$  where  $x_1 > 0$  then  $c = m/(m + 1)$ .*

Let  $\xi(v) = (q, -nq, -q, nq)$  where  $q > 0$  and  $n > 0$ . Let  $\gamma_- : [0, 3] \rightarrow CM$  such that  $\gamma_-(0) = v(0)$ ,  $\gamma_-(1) = \xi(v)$ , and  $\gamma_-(2) = (nq, q, nq, q)$ .

**PROPOSITION 3.8.** *For  $s$  sufficiently small, the orbit of  $v(s)$  lies in an  $\varepsilon$ -tube about  $\gamma_-$  over the interval  $[0, t^*]$ . Furthermore, if  $P_1^*(w(\xi(v), \eta_-(v); c)) = (x_1, 0)$ ,  $x_1 > 0$  then  $c = n/(n + 1)$ .*

The proof of Proposition 3.8 is similar to that of Propositions 3.6 and 3.7. Direct calculation gives

$$P_2^*(w(\xi(u), \eta_+(u); m/m + 1)) = \frac{1}{\sqrt{2}} \frac{1}{1 + m^2} (m^2 - 1, -2m)$$

$$P_2^*(w(\xi(v), \eta_-(v); n/n + 1)) = \frac{1}{\sqrt{2}} \frac{1}{1 + n^2} (n^2 - 1, 2n).$$

**DEFINITION 3.9.**  $\alpha(0) \in [-\pi, 0]$  and  $\beta(0) \in [0, \pi]$  are given by

$$\cos(\alpha(0)) = \frac{P_2^* w}{\|P_2^* w\|} \cdot (1, 0) = \frac{m^2 - 1}{m^2 + 1}$$

$$\cos(\beta(0)) = \frac{P_2^* w}{\|P_2^* w\|} \cdot (1, 0) = \frac{n^2 - 1}{n^2 + 1}.$$

Using this definition one has that  $\alpha$  and  $\beta$  are continuous on  $[0, 1]$  and hence that  $\Psi$  is continuous on  $[0, 1]$ .

PROPOSITION 3.10.  $\Psi(0) > \pi$ .

*Proof.* Let

$$I = \frac{P_2^*(w(\xi(u), \eta_+(u); m/(m+1)))}{\|P_2^*(w(\xi(u), \eta_+(u); m/(m+1)))\|}$$

and

$$I^* = \frac{P_2^*(w(\xi(v), \eta_-(v); n/(n+1)))}{\|P_2^*(w(\xi(v), \eta_-(v); n/(n+1)))\|}.$$

Then  $I = (1/(m^2 + 1))(m^2 - 1, -2m)$  and  $I^* = (1/(n^2 + 1))(n^2 - 1, 2n)$ . Since  $m, n > 0$  it is clear that  $I$  lies in quadrants III and IV of the plane, and  $I^*$  lies in quadrants I and II of the plane. If it can be shown that  $I^*$  lies to the left of  $-I$  then clearly  $\Psi(0) > \pi$ . Let  $(a, b) = -I - I^*$ . Showing that  $I^*$  lies to the left of  $-I$  is equivalent to showing that  $a > 0$ . Now,

$$\begin{aligned} a &= (1 - m^2)/(1 + m^2) + (1 - n^2)/(1 + n^2) \\ &= 2(1 - m^2n^2)/(m^2 + 1)(n^2 + 1), \end{aligned}$$

but Assumption 3 implies that  $mn < 1$ , and hence  $a > 0$ . ■

Let us for the moment consider the case  $a > 1$  in (2.12) and how it differs from what we have just done for  $a = 1$ . The  $x_1$ -axis has been singled out as a reference line in Assumptions 1, 2, and 4. When  $a = 1$  this is not a restriction since  $V$  is locally symmetric about the origin. If  $a > 1$  then the results one obtains will depend upon the reference line chosen. Assumption 2(a) works for  $a = 1$  since all elements of  $S^s$  are critical points. If  $a > 1$  then the results will change depending on whether  $u(0) \cdot t$  converges to  $\pm(1/\sqrt{2})(1, 0, -1, 0)$  or  $\pm(1/\sqrt{1+a^2})(0, 1, 0, -a)$ . Finally, we were able to give a sharp estimate for Assumption 3 (and hence Theorem 3.5) because we knew how the orbits  $u(s)$  converged to  $\gamma_{\pm}(u(s))$  as  $s \rightarrow 0$ . If  $a > 1$  then the limit of the crossing angles will be sensitive to  $m, n$ , and  $a$ . Therefore, while one can perform the same type of analysis for  $a > 1$ , the arguments will have to be more delicate or the resulting theorems less precise.

DEFINITION 3.11. Let  $M(h) = \{z \mid H(z) = h\}$ . Let  $u(h): [0, 1] \rightarrow ET(h)$  and  $v(h): [0, 1] \rightarrow EB(h)$  be parametrizations of portions of  $ET(h)$  and  $EB(h)$ , respectively.

The results obtained up to this point have been proven only on the invariant surface  $M$ . This is due to the fact that any orbit which lies in  $M(h)$ , where  $h \neq 0$ , is bounded away from the origin in  $R^4$ . Therefore, one cannot expect that the flow on  $CT$  would provide a reasonable approximation to the orbits on  $M(h)$ . However, having found results for  $M(0)$ , slightly stronger conditions on  $V$ , as well as the fact that  $V$  is continuous, should allow one to conclude that similar results hold for  $M(h)$ , as long as  $h$  is chosen sufficiently close to 0.

- ASSUMPTION 6. (a)  $u$  and  $v$  are continuous on  $[h^-, h^+] \times [0, 1]$ .  
 (b) For all  $s \in (0, 1]$ ,  $\lambda(u(h, s))$  and  $\lambda(v(h, s))$  exist.  
 (c)  $P_1(u(h, s) \cdot \lambda(u(h, s))) = P_1(v(h, s) \cdot \lambda(v(h, s)))$ .

DEFINITION 3.12.  $\alpha(h, s) \in [-\pi, 0]$  and  $\beta(h, s) \in [0, \pi]$  are defined on  $[h^-, h^+] \times (0, 1]$  by

$$\begin{aligned}\cos(\alpha(h, s)) &= \frac{\langle P_2(u(h, s) \cdot \lambda(u(h, s))), (1, 0) \rangle}{\|P_2(u(h, s) \cdot \lambda(u(h, s)))\|} \\ \cos(\beta(h, s)) &= \frac{\langle P_2(v(h, s) \cdot \lambda(v(h, s))), (1, 0) \rangle}{\|P_2(v(h, s) \cdot \lambda(v(h, s)))\|} \\ \Psi(h, s) &= \beta(h, s) - \alpha(h, s).\end{aligned}$$

Notice that  $\alpha$ ,  $\beta$ , and, hence,  $\Psi$  are continuous on  $[h^-, h^+] \times (0, 1]$ .

THEOREM 3.13. Given Assumptions 1–6 and given  $|h^+|$  and  $|h^-|$  sufficiently small, for each  $h \in [h^-, h^+]$  there exists at least one periodic orbit  $\Pi_3(h)$ , which lies in  $M(h)$  and intersects  $ET(h)$  and  $EB(h)$ .

*Proof.* By Theorem 3.5 there exists an  $s^*$  such that  $\Psi(0, s^*) = \pi$ . Furthermore, by Proposition 3.10,  $\Psi(0, 0) > \pi$ . Thus, there exists an  $\tilde{s} \in (0, s^*)$  such that  $\Psi(0, \tilde{s}) > \pi$ . Since  $\Psi$  is continuous in a neighborhood of  $(0, \tilde{s})$ , there exists  $\delta_0 > 0$  such that  $\Psi(\varepsilon, \tilde{s}) > \pi$  for  $\varepsilon \in [-\delta_0, \delta_0]$ . Similarly, by Assumption 5  $\Psi(0, 1) < \pi$ , hence there exists  $\delta_1 > 0$  such that  $\Psi(\varepsilon, 1) < \pi$  for  $\varepsilon \in [-\delta_1, \delta_1]$ .

Let  $h^- = \max(-\delta_0, -\delta_1)$  and  $h^+ = \min(\delta_0, \delta_1)$ . Then  $h \in [h^-, h^+]$  implies that there exists  $s^*(h)$  such that  $\Psi(h, s^*(h)) = \pi$ . ■

From now on  $h^\pm$  will be chosen as in Theorem 3.13.

ASSUMPTION 7. If  $s_1 < s_2$  then  $|\alpha(h, s_1)| > |\alpha(h, s_2)|$  and  $|\beta(h, s_1)| > |\beta(h, s_2)|$ .

LEMMA 3.14. *Assumptions 1–7 ensure that  $\Pi_3(h)$  is unique for a given  $h \in [h^-, h^+]$ .*

DEFINITION 3.15. Let  $s_0(h) \in (0, \tilde{s}(h)]$  such that

$$u(h, s_0(h)) \cdot [0, \lambda(u(h, s_0(h)))] \cap u(h, \tilde{s}(h)) \cdot [0, \lambda(u(h, \tilde{s}(h)))] \neq \phi.$$

But, if  $s \in (0, s_0(h))$  then

$$u(h, s) \cdot [0, \lambda(u(h, s))] \cap u(h, \tilde{s}(h)) \cdot [0, \lambda(u(h, \tilde{s}(h)))] = \phi.$$

Let  $x(h, s)$  be the  $x_1$ -coordinate of  $P_1(u(h, s)) \cdot \lambda(u(h, s))$ . Let  $s_1(h) \in (0, \tilde{s}(h)]$  such that  $x(h, s_1(h)) = x(h, \tilde{s}(h))$  and if  $s \in (0, s_1(h))$  then  $x(h, s) < x(h, s_1(h))$ .

LEMMA 3.16.  $x(h, s_0(h)) \leq x(h, \tilde{s}(h))$  and if  $s < s_0$  then  $x(h, s) < x(h, \tilde{s}(h))$ .

PROPOSITION 3.17.  $s_0(h) = s_1(h) = \tilde{s}(h)$ .

*Proof.* In what follows,  $h$  is considered fixed and hence, to simplify the notation, is suppressed. By definition  $s_0 \leq \tilde{s}$ . So assume  $s_0 < \tilde{s}$ . Furthermore assume  $x(s_0) = x(\tilde{s})$ . By Assumption 7  $|\alpha(s_0)| > |\alpha(\tilde{s})|$ , hence one must be in the situation of Fig. 5. But by continuity of the flow, there exists  $s'' < s_0$  such that

$$u(s'') \cdot [0, \lambda(u(s''))] \cap u(\tilde{s}) \cdot [0, \lambda(u(\tilde{s}))] \neq \phi,$$

a contradiction. Thus  $x(s_0) < x(\tilde{s})$ . Now either  $u(s_0) \cdot [0, \lambda(u(s_0))]$  intersects  $u(\tilde{s}) \cdot [0, \lambda(u(\tilde{s}))]$  topologically transversally or not; i.e., tangentially but not topologically transversally. The former cannot happen since this forces the existence of  $s''$  as above. But if the intersection is tangential then

$$u(s_0) \cdot [0, \lambda(u(s_0))] = u(\tilde{s}) \cdot [0, \lambda(u(\tilde{s}))].$$

Therefore,  $s_0 = \tilde{s}$ .

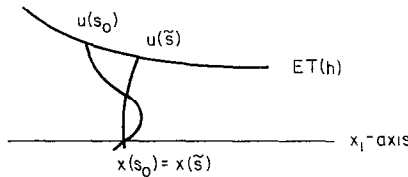


FIGURE 5

Thus either  $0 < s_1 < s_0 = \hat{s}$  or  $s_1 = s_0 = \hat{s}$ . Assume the former then  $x(s_1) < x(\hat{s})$ , a contradiction. ■

COROLLARY 3.18. *Given Assumptions 1–7:*

- (a)  $u(h, s) \cdot [0, \lambda(u(h, s))] \cap u(h, S) \cdot [0, \lambda(u(h, S))] = \emptyset$  if  $s \neq S$ .
- (b)  $v(h, s) \cdot [0, \lambda(v(h, s))] \cap v(h, S) \cdot [0, \lambda(v(h, S))] = \emptyset$  if  $s \neq S$ .

*Proof.* The machinery developed starting with Lemma 3.14 proves (a). The proof for (b) is similar. ■

THEOREM 3.19. *Given Assumptions 1–7, for  $h \in [h^-, h^+]$ ,  $\Pi_3(h)$  is an isolated periodic orbit in  $M(h)$ ; i.e., there exists a neighborhood  $U$  of  $M(h)$  such that  $M(h)$  is the maximal invariant set in  $U$ .*

*Proof.* First one constructs what will be the isolating set. Since  $s^*(h)$  is the unique solution to  $\Psi(s, h) = \pi$ , it must be that for  $0 < s_1 < s^* < s_2 < 1$ , one has  $\Psi(h, s_1(h)) > \pi > \Psi(h, s_2(h))$ .

The orbits  $P_1(u(s_i(h)) \cdot [0, \lambda(u(s_i(h))))$  and  $P_1(v(s_i(h)) \cdot [0, \lambda(v(s_i(h))))$  for  $i = 1, 2$  can be used to define a compact region, denoted  $N_{12}(h)$ , contained in the set  $\{x \mid V(x) \leq h\}$  (see Fig. 6). Let

$$PN_{12}(h) = P_1^{-1}(h)(N_{12}(h)).$$

To see that  $PN_{12}(h)$  is an isolating neighborhood of  $\Pi_3(h)$ , one must show that the flow has no internal tangencies on  $PN_{12}(h)$ . By construction, the only tangential orbits are those which pass through  $u(s_i(h))$  or  $v(s_i(h))$  for  $i = 1, 2$ . However, the conditions on  $\alpha(s_i(h))$  or  $\beta(s_i(h))$  force the orbits to leave in positive and negative time at the respective points

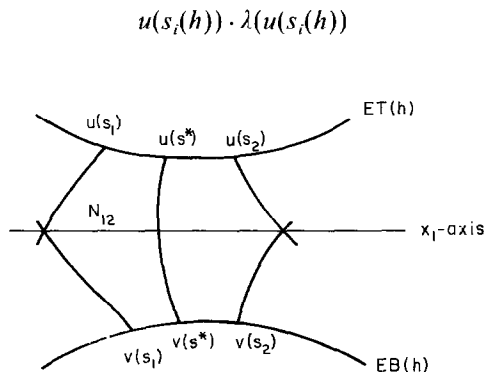


FIGURE 6

and

$$v(s_i(h)) \cdot \lambda(v(s_i(h))).$$

Thus there are no internal tangencies.

One now needs to show that  $\Pi_3(h)$  is the maximal invariant set in  $PN_{12}(h)$ . Assume not, i.e., assume that there exists some other orbit  $\Pi' \neq \Pi_3(h)$  which is contained in  $PN_{12}(h)$  for all time. Given  $s_j(h)$  and  $s_k(h)$  such that

$$s_1(h) \leq s_j(h) \leq s^*(h) \leq s_k(h) \leq s_2(h),$$

construct  $PN_{jk}(h) \subset M(h)$  in the same manner that  $PN_{12}(h)$  was constructed from  $s_1$  and  $s_2$ . Let

$$PN(h) = \bigcap_{j,k} \{PN_{jk}(h) \mid \Pi' \subset PN_{jk}(h)\}.$$

By Corollary 3.18, there exists  $s_{j*}(h)$  and  $s_{k*}(h)$  such that  $PN(h) = PN_{j*k*}(h)$ . Thus  $\Pi'$  must have a point of tangency with  $PN(h)$ . But since  $\Pi' \neq \Pi_3$ , this means that  $\Pi'$  must leave  $PN(h)$ , a contradiction. ■

**DEFINITION 3.20.** A line (segment)  $L$  is a *gradient line (segment)* of  $V$ , if for every  $x \in L$  such that  $\nabla V(x) \neq 0$ , one has that  $\nabla V(x)$  is parallel to  $L$ .

**ASSUMPTION 8.** *There exist gradient line segments  $L_i$ ,  $i = 1, 2$  of  $V$  which intersect the origin. Furthermore,  $L_1$  intersects  $u(h, 0)$  and  $L_2$  intersects  $v(h, 0)$  for all  $h \in [h^-, h^+]$ .*

The following proposition is obvious.

**PROPOSITION 3.21.** *If  $h \in [h^-, 0]$  then there exist bounded orbits  $\Pi_i(h)$  such that*

- (a)  $P_1(\Pi_1(h)) \subset L_1$  and intersects both  $EO(h)$  and  $ET(h)$ .
- (b)  $P_1(\Pi_2(h)) \subset L_2$  and intersects both  $EO(h)$  and  $EB(h)$ .

*Furthermore, if  $h < 0$  then  $\Pi_i(h)$  is a periodic orbit and if  $h = 0$  then  $\Pi_i(h)$  is a homoclinic orbit with the origin as the critical point.*

Let  $V: (0, 90] \times \mathbb{R}^2 \rightarrow \mathbb{R}$  be continuous,  $\Phi \in (0, 90]$ , and write  $V_\Phi(x) = V(\Phi, x)$ . Assume that for fixed  $\Phi$ ,  $V_\Phi$  satisfies Assumptions 1, 2, and 4–8 where  $\Phi$  equals the angle in degrees between  $L_1$  and  $L_2$ . Again fixing  $\Phi$ , define  $\alpha_\Phi(s)$ ,  $\beta_\Phi(s)$ , and  $\Psi_\Phi(s)$  as before for the potential function  $V_\Phi$ .

PROPOSITION 3.22. Given  $V_\phi$  as above,  $\Psi_\phi(s)$  is a continuous function on  $(0, 90] \times (0, 1]$ . In addition, for fixed  $\Phi$ ,  $\Psi_\phi$  is continuous on  $[0, 1]$ .

*Proof.* The fact that  $\Psi_\phi$  is continuous on  $[0, 1]$  follows from the construction of  $\Psi_\phi$  and the definition of  $\Psi_\phi(0)$ .

For fixed  $\Phi$ , one has the partial parametrization of  $ET(0)$  and  $EB(0)$  corresponding to  $V_\phi$ . Denote these by  $u_\phi$  and  $v_\phi$ . Since  $V$  is continuous on  $(0, 90] \times \mathbb{R}^2$ , one can choose  $u_\phi$  and  $v_\phi$  to be continuous on  $(0, 90] \times [0, 1]$ . Showing that  $\Psi$  is continuous on  $(0, 90] \times (0, 1]$  is equivalent to showing that  $\alpha$  and  $\beta$  are continuous on this region. Because  $\alpha$  and  $\beta$  are similar, it is enough to show that  $\alpha$  is continuous:

$$\alpha_\phi(s) = \cos^{-1} \frac{\langle P_2(u_\phi(s) \cdot \lambda(u_\phi(s))), (1, 0) \rangle}{\|P_2(u_\phi(s) \cdot \lambda(u_\phi(s)))\|}.$$

Hence, it is sufficient to show that  $u_\phi(s) \cdot \lambda(u_\phi(s))$  is continuous on  $(0, 90] \times (0, 1]$ . Because  $u$  is continuous, given  $\varepsilon > 0$ , there exists  $\delta > 0$ , such that if  $\|(\Phi, s) - (\Phi_0, s)\| < \delta$  then  $\|u_\phi(s) - u_{\Phi_0}(s)\| < \varepsilon$ . Now by the standard theorems on continuity with respect to initial conditions for solutions of ODE's one has that  $u_\phi(s) \cdot \lambda(u_\phi(s))$  is continuous. ■

THEOREM 3.23. Let  $V$  be as above. Let  $s^*(\Phi)$  be the unique solution to  $\Psi(\Phi, s) = \pi$ . Then  $\lim_{\Phi \rightarrow 90} s^*(\Phi) = 0$ .

*Proof.* Reviewing the proof of Proposition 3.10, one sees that  $\Psi(90, 0) = \pi$ . Hence by Assumption 7,  $\Psi(90, s) < \pi$  for all  $s > 0$ . Let  $\{\Phi_n\}$  be a sequence such that  $\Phi_n \rightarrow 90$  and  $s^*(\Phi_n) \rightarrow s_0$ . By continuity of dependence on parameters for ODE's, one has that  $\Phi(90, s_0) = \pi$ . Hence  $s_0 = 0$ . ■

It was shown earlier (Theorem 3.5) that if the angle  $\Phi$  between the projection of the two homoclinic orbits  $\Pi_i(0)$ ,  $i = 1, 2$ , onto  $x$ -space is less than  $90^\circ$ , then there exists a unique periodic orbit  $\Pi_3(0)$ . Theorem 3.23 says that as this angle goes to  $90^\circ$  from below, the periodic orbit  $\Pi_3(0)$  collapses onto the two homoclinic orbits,  $\Pi_1(0)$  and  $\Pi_2(0)$ .

#### 4. CROSSING ORBITS

Let  $V$  be a potential functions as in the previous chapters satisfying Assumptions 1–8. Furthermore, for the sake of simplicity assume that the positive  $x_1$ -axis is a gradient line of  $V$ . Let  $P_1(h)(\Pi_i(h)) = \pi_i(h)$  for  $i = 1, 2, 3$ . Let  $J(h)$ ,  $h \in [h^-, h^+]$  be the compact region in  $\mathbb{R}^2$  with boundaries given by  $EO(h)$ ,  $ET(h)$ ,  $EB(h)$ ,  $L_i$  for  $i = 1, 2$ , and  $\pi_3(h)$ . (see Fig. 7).

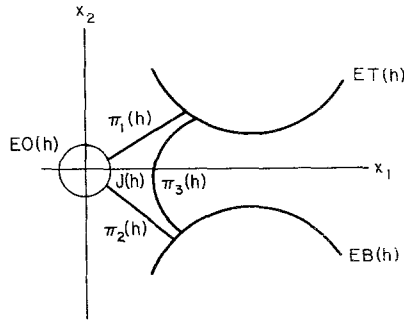


FIGURE 7

Let  $\mathcal{J}(h) = P_1^{-1}(h) J(h)$ . Given  $z \in \mathbb{R}^4$  define  $\theta(z) = z \cdot \mathbb{R}$  and  $\theta_i(z) = P_i(\theta(z))$ ,  $i = 1, 2$ .

**DEFINITION 4.1.** Let  $H(z) = h$ . Assume that there exists  $t_0$  and  $t_1$  such that  $t_0 < t_1$ ,  $P_1(z \cdot t_0) \in \pi_i(h)$ ,  $P_1(z \cdot t_1) \in \pi_j(h)$ , and  $z \cdot [t_0, t_1] \in \mathcal{J}(h)$ . Then one denotes  $z$  by  $z_{ij}(h)$  and  $z_{ij}(h)$  is called a *crossing orbit from  $\pi_i(h)$  to  $\pi_j(h)$* .

In use  $z_{ij}(h)$  will be taken to mean both an orbit with the above property and a generic point on such an orbit. Notice that  $\theta_1(z_{ij}(h))$  intersects  $\pi_i(h)$  and  $\pi_j(h)$  transversally. In particular, the orbit originates outside of  $J(h)$ , enters  $J(h)$  via  $\pi_i(h)$ , and exits  $J(h)$  via  $\pi_j(h)$ .

**THEOREM 4.2.** *There exists  $z_{ij}(0)$  for  $i, j = 1, 2, 3$ , (except  $i = j = 3$ ). Furthermore, there exists  $\tilde{v}(0) \in EB(0)$  and  $\tilde{u}(0) \in ET(0)$  so that  $\tilde{v}(0) = z_{11}(0)$  and  $\tilde{u}(0) = z_{22}(0)$ .*

*Proof.* Let  $\tilde{u} = u(0, \tilde{s})$ . Then for  $\tilde{s}$  sufficiently close to 0, the orbit of lies in an  $\varepsilon$ -tube about  $\gamma_+$  over the interval  $[0, t^*]$  (see Definitions 2.17 and 2.18). Hence there exists  $t_1 > 0$  such that  $P_1(\tilde{u} \cdot t_1) \in \pi_2(0)$  (see Fig. 8a). Since the system is reversible, letting  $t_0 = -t_1$ , one has  $P_1(\tilde{u} \cdot t_0) \in \pi_2(0)$ . Clearly,  $P_1(\tilde{u} \cdot [t_0, t_1]) \subset J(0)$ . Therefore,  $\tilde{u} = z_{22}(0)$ .

The proof that there exists  $\tilde{v}(0) = z_{11}(0)$  is similar.

To find  $z_{3j}$ , where  $j = 1, 2$ , notice that because the positive  $x_1$ -axis is gradient line there exists  $z \in \mathcal{J}(0)$ , such that  $P_1(z) = (x_1, 0)$ , where  $x_1 > 0$  and  $z$  is on the stable manifold of the origin. By Definition 2.5, one has the  $\lim_{t \rightarrow \infty} z \cdot t = (0, \xi)$  where  $\xi = (\xi_1, 0, -\xi_1, 0)$  and hence one can define  $\gamma_{\pm}(z)$  as usual. Now choose  $\tilde{z}$  close to  $z$ , so that the orbit of  $\tilde{z}$  lies in an  $\varepsilon$ -tube about  $\gamma_+(z)$  on the interval  $[0, t^*]$ , but so that  $\tilde{z}$  is not on the stable manifold. Then  $\tilde{z} = z_{32}$ . Similarly, choosing  $\tilde{z}$  close to  $z$ , again, not on the stable manifold, but lying in an  $\varepsilon$ -tube about  $\gamma_-$ , gives  $z_{31}$  (see Fig. 8b). The reversibility of the system implies the existence of  $z_{13}$  and  $z_{23}$ .



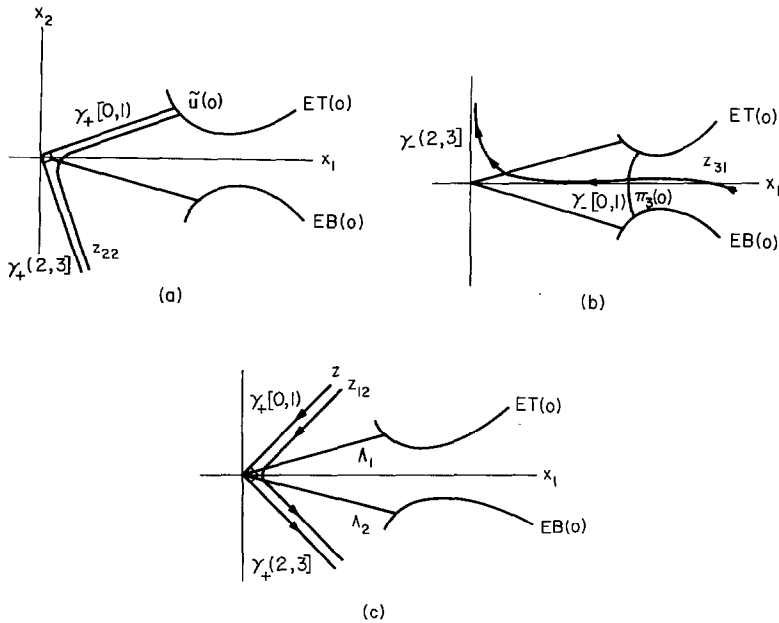


FIGURE 8

Recall Assumption 8. Let  $m_i$  equal the slope of  $L_i$  where  $i=1, 2$ . By Assumption 3,  $(-1/m_1) < m_2$ . So choose  $z$  on the stable manifold of the origin so that  $P_1(z) = (x_1, mx_1)$  where  $m > m_1$  and  $-1/m < m_2$  (see Fig. 8c). As before, construct the  $\gamma_+$  curve corresponding to  $z$ . Then there exists  $\tilde{z}$  close to  $z$ , so that the orbit of  $\tilde{z}$  lies in a  $[0, 3]$   $\varepsilon$ -tube about  $\gamma_+$ . Since  $z \in \mathcal{J}(0)$ ,  $\tilde{z}$  can be chosen such that  $\tilde{z} \in \mathcal{J}(0)$ . Now one readily checks that there exists  $t_0 < t_1$  such that  $P_1(\tilde{z} \cdot t_0) \in \pi_1(0)$ ,  $P_1(\tilde{z} \cdot t) \in \pi_2(0)$ , and  $\tilde{z} \cdot [t_0, t_1] \subset \mathcal{J}(0)$ . Thus,  $\tilde{z} = z_{12}(0)$ . Again, the reversibility of the system implies the existence of  $z_{21}$ . ■

**THEOREM 4.3.** *Let  $h \in [h^-, h^+]$ . Then there exists  $z_{ij}(h)$  where  $i, j=1, 2, 3$  (except  $i=j=3$ ). If  $h \in [h^-, 0)$  then  $z_{33}(h)$  exists. Furthermore, one can choose  $\tilde{v}(h) \subset EB(h)$  and  $\tilde{u}(h) \subset ET(h)$  so that  $\tilde{v}(h) = z_{11}(h)$  and  $\tilde{u}(h) = z_{22}(h)$ .*

*Proof.* If  $h < 0$  then choose  $z$  so that  $H(z) = h$ ,  $P_1(z) = (x_1, 0)$  and  $x_1 > 0$  and  $P_2(z) = (y_1, 0)$  where  $y_1 < 0$ . Then  $z = z_{33}(h)$ .

Theorem 4.2 gives the existence of  $z_{ij}(0)$ . Having chosen a particular  $z_{ij}(0)$ , notice that  $\theta(z_{ij}(0))$  is bounded away from the origin and that  $\theta_1(z_{ij}(0))$  intersects  $\pi_1(0)$  and  $\pi_j(0)$  transversally. Thus for  $|h^-|$  and  $|h^+|$

sufficiently small, one can invoke the continuity of the flow to ensure that there exists  $z_{ij}(h)$ .

The existence of  $\tilde{v}(h) = z_{11}(h)$  and  $\tilde{u}(h) = z_{22}(h)$  also follows from continuity. ■

**THEOREM 4.4.** *For  $\tilde{x}_1 > 0$ , but sufficiently small it is possible to choose  $z_{ij}(0)$ , for  $i, j = 1, 2, 3$  (except  $i = j = 3$ ), such that  $(\tilde{x}_1, 0) \in \theta_1(z_{ij}(0))$ .*

*Proof.* Recall the proof of Theorem 4.2. In each case,  $z_{ij}$  was shown to exist by choosing a  $\tilde{z}$  whose orbit lay inside an appropriate  $\gamma_{\pm} \varepsilon$ -tube. But each such  $\varepsilon$ -tube contains orbits which lie arbitrarily close to the origin and hence the positive  $x_1$ -axis intersected with each  $\varepsilon$ -tube gives an open interval of the form  $(0, a)$ . Choosing the minimum of these five  $a$ 's gives an interval  $(0, \tilde{a})$  such that if  $\tilde{x}_1 \in (0, \tilde{a})$  then there exists  $z_{ij}$  such that  $(\tilde{x}_1, 0) \in \theta_1(z_{ij})$ . ■

**COROLLARY 4.5.** *For  $h \in [h^-, h^+]$ , one can choose  $z_{ij}(h)$ , where  $i, j = 1, 2, 3$  (except for  $i = j = 3$ ), such that there exists  $\tilde{x}_1(h) > 0$  with  $(\tilde{x}_1(h), 0) \in \theta_1(z_{ij}(h))$ . If  $h \in [h^-, 0)$  then the same is true for  $i = j = 3$ .*

From now on it is assumed that  $z_{ij}(h)$  is chosen in this manner.

In what follows the techniques of Rod [7] are used extensively. It is assumed that the reader is familiar with [7] or has a copy readily available. A slight modification of his work is necessary since  $\Pi_i(0)$ ,  $i = 1, 2$  are homoclinic orbits rather than periodic orbits. Thus we are forced to regenerate his definitions and lemmas in this different setting. We shall use the notation of [7] and refer to the proofs therein whenever possible.

The orbits  $\theta(z_{ij}(h))$  will be used to divide  $\mathcal{J}(h)$  into three regions, which in turn can be used to classify the solutions of (2.12) which intersect  $\mathcal{J}(h)$ .

**Notation 4.6** For  $h \in [h^-, h^+]$ , let

$$\bar{D}_1(h) = \{x = (x_1, x_2) \mid x \in \theta_1(\tilde{v}(h)) \text{ and } x_2 \leq 0\} \subset J(h)$$

$$\bar{D}_2(h) = \{x = (x_1, x_2) \mid x \in \theta_1(\tilde{u}(h)) \text{ and } x_2 \geq 0\} \subset J(h)$$

$$\bar{D}_3(h) = \{x = (x_1, x_2) \mid x \in \theta_1(z_{33}(h)) \text{ and } x_1 \leq \tilde{x}_1(h)\}.$$

Strictly speaking  $\bar{D}_3(h)$  cannot be defined as above if  $h \geq 0$ . So in this case let

$$\bar{D}_3(h) = \{(x_1, 0) \mid 0 \leq x_1 \leq \tilde{x}_1(h)\}$$

$$\bar{\Sigma}_i(h) = \bar{D}_j(h) \cup \bar{D}_k(h) \quad \text{for } i \neq j \neq k \neq i$$

$$\Sigma_i(h) = P_1^{-1}(h)(\bar{\Sigma}_i(h)), \quad i = 1, 2, 3$$

$$D_i(h) = P_1^{-1}(h)(\bar{D}_i(h)).$$

For  $h \leq 0$ , set

$$\bar{\Sigma}_3^* = \{x = (x_1, x_2) \mid \text{if } x_2 \geq 0 \text{ then } x \in \theta_1(u(h, 1)), \\ \text{and if } x_2 \leq 0 \text{ then } x \in \theta_1(v(h, 1))\}.$$

Referring to Fig. 9, let  $\bar{\Sigma}_i^*(h)$ ,  $i=1, 2$ , be the curves shown (see Assumption 9 below). In particular,  $\bar{\Sigma}_1^*(h)$  connects  $ET(h)$  with  $EO(h)$  and  $\bar{\Sigma}_2^*(h)$  connects  $EB(h)$  with  $EO(h)$ . Let

$$\Sigma_i^*(h) = P_1^{-1}(h)(\bar{\Sigma}_i^*(h)).$$

$\bar{R}_i(h)$  is the compact region in  $J(h)$  bounded by  $\bar{\Sigma}_i(h)$  and  $\bar{\Sigma}_i^*(h)$  for  $i=1, 2, 3$ , segments of  $ET(u)$  and  $EB(h)$ , and the  $x_1$ -axis (see Fig. 9). Set

$$R_i(h) = P_1^{-1}(h)(\bar{R}_i(h)).$$

Theorem 3.19 guarantees that  $R_3(h)$  is an isolating neighborhood for  $\Pi_3(h)$ ,  $h \in [h^-, h^+]$ .

**ASSUMPTION 9.** *There exists  $\bar{\Sigma}_i^*(h)$  for  $i=1, 2$ , so that if  $h < 0$  then  $R_i(h)$  is an isolating block for  $\Pi_i(h)$ . If  $h=0$  then  $R_i(0)$  is as in Fig. 10, in which case, it is assumed that the only bounded orbits in  $R_i(0)$  are  $\Pi_i(0)$  and the fixed point at the origin. Furthermore, any orbit which is tangent to  $\Sigma_i(0)$  or  $\Sigma_i^*(0)$  lies either on the stable manifold, or on the unstable manifold of the origin, or leaves  $R_i(0)$  in forward and backward time without entering the interior of  $R_i(0)$ .*

The importance of this assumption is that  $R_1(h)$  and  $R_2(h)$  are isolating blocks for  $\Pi_1(h)$  and  $\Pi_2(h)$ . For conditions on  $V$  which induce the

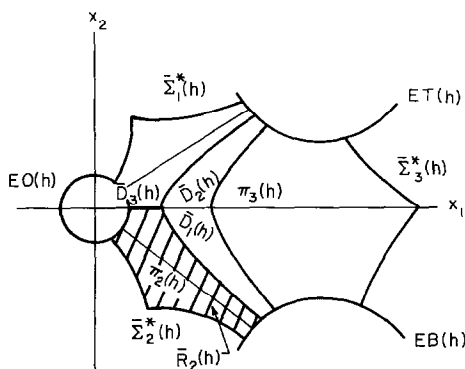


FIGURE 9

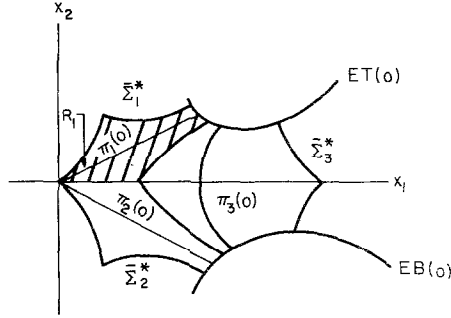


FIGURE 10

existence of such isolating neighborhoods the reader is referred to Churchill, Pecelli, and Rod [1].

LEMMA 4.7. (Rod [7, Lemma 3.1]). *Let  $i = 1, 2, 3$ ,  $h \leq 0$ . Then*

- (a)  $D_i(h)$  is a closed topological two-disk, and
- (b) each  $\Sigma_i$  and each  $\Sigma_i^*$  is a topological two-sphere.

DEFINITION 4.8. For  $i = 1, 2, 3$ ,  $h \leq 0$  let

$$b_i^+(h) = \{z \in \Sigma_i(h) \mid \text{there exists } \varepsilon > 0 \text{ with } z \cdot (0, \varepsilon] \subset \text{int } R_i(h)\}$$

$$b_i^-(h) = \{z \in \Sigma_i(h) \mid \text{there exists } \varepsilon > 0 \text{ with } z \cdot [-\varepsilon, 0) \subset \text{int } R_i(h)\}$$

$$T_i(h) = \Sigma_i(h) / \{b_i^+(h) \cup b_i^-(h)\}.$$

LEMMA 4.9. For  $h \leq 0$ ,  $i = 1, 2, 3$ ,

- (a) the  $b_i^\pm$  are disjoint open hemispheres in  $\Sigma_i$  transverse to the flow with  $b_i^- = \{(x, -y) \mid (x, y) \in b_i^+\}$ .
- (b) The tangency set,  $T_i$ , is homomorphic to a circle.
- (c) The orbits through points of  $T_i$  "bounce off the region  $R_i$  to the outside" (except when  $h = 0$  and  $i = 1, 2$ ).

*Proof.* For  $h < 0$  or  $i = 3$  see [7, Lemma 3.3]. So consider  $i = 1$ ,  $h = 0$ . By Lemma 4.7,  $D_2$  and  $D_3$  can be represented as in Fig. 11 where  $P_1(\partial D_i) = (\hat{x}_1, 0)$  and concentric circles project to single points in  $R^2$  under  $P_1$ . Notice the orientation chosen for the  $y$  values. The dark lines represent the tangencies to  $D_2$  and  $D_3$  and thus  $T_1(0)$  is homeomorphic to  $S^1$ .

The proof for  $T_2(0)$  is similar except one considers  $D_1$  and  $D_3$ . ■

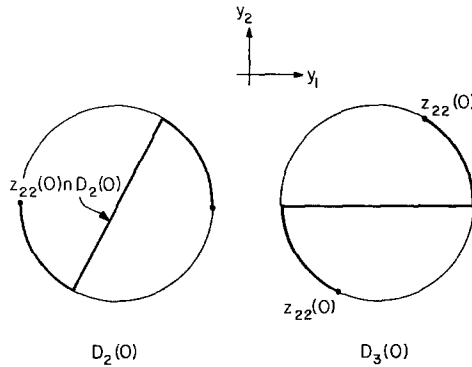


FIGURE 11

DEFINITION 4.10. For  $i = 1, 2, 3$ ,  $h \leq 0$ , and  $z \in b_i^+(h)$  set

$$\sigma_i^+(h, z) = \inf\{t > 0 \mid z \cdot t \in b_i^-(h)\};$$

for  $z \in b_i^-(h)$  set

$$\sigma_i^-(h, z) = \sup\{t < 0 \mid z \cdot t \in b_i^+(h)\},$$

provided that the inf and sup exist. Define  $\varphi_i^\pm(h): b_i^\mp(h) \rightarrow b_i^\mp(h)$  by  $\varphi_i^\pm(h)(z) = z \cdot \sigma_i^\pm(h, z)$  where the domain of  $\varphi_i^\pm(h)$  is the same as that for  $\sigma_i^\pm(h)$ .

LEMMA 4.11 (Rod [7, Lemma 3.4]). For  $h \in [h^-, 0]$ ,  $i = 1, 2, 3$ ,  $\sigma_i^\pm(h)$  is continuous where defined with the domain being an open subset of  $b_i^\pm(h)$ .  $\varphi_i^\pm(h)$  is a homeomorphism from domain to range with inverse  $\varphi_i^\mp(h)$ .

DEFINITION 4.12. For  $h \in [h^-, 0]$ ,  $i = 1, 2, 3$ , define

$$T_i^+(h) = \{z \in b_i^+ \mid z \cdot (0, \infty) \cap \Sigma_i^* \neq \emptyset \text{ and } z \cdot (0, \infty) \cap \Sigma_i = \emptyset\}$$

$$T_i^-(h) = \{z \in b_i^- \mid z \cdot (-\infty, 0) \cap \Sigma_i^* \neq \emptyset \text{ and } z \cdot (-\infty, 0) \cap \Sigma_i = \emptyset\}.$$

Notice that  $T_i^\pm(h)$  consists of those points whose orbits pass from  $\Sigma_i$  through  $R_i$  and leave via  $\Sigma_i^*$  in  $\pm$  time.

DEFINITION 4.13. Let  $NT_i^\pm(h)$ ,  $i = 1, 2, 3$ ,  $h \in [h^-, 0]$ , be the domain in  $b_i^\pm(h)$  of the mapping  $\varphi_i^\pm(h)$ .

DEFINITION 4.14. For  $i = 1, 2, 3$ ,  $h \in [h^-, 0]$ , let

$$a_i^+(h) = \{z \in b_i^+(h) \mid z \cdot [0, \infty) \subset R_i\}$$

$$a_i^-(h) = \{z \in b_i^-(h) \mid z \cdot [-\infty, 0] \subset R_i\}.$$

In addition for  $i = 1, 2$ , the origin is included in  $a_i^\pm(0)$ .

It should be clear that

LEMMA 4.15. For  $i = 1, 2, 3$ ,  $h \in [h^-, 0)$ ,  $T_i^\pm(h)$ ,  $NT_i^\pm(h)$ , and  $a_i^\pm(h)$  are disjoint. This is also true for  $i = 3$  and  $h = 0$ . If  $i = 1, 2$ , and  $h = 0$ , then this holds except that  $a_i^+(0) \cap a_i^-(0)$  is the origin.

It should also be clear that any orbit in  $\mathcal{J}(h)$  which passes through  $T_i^\pm(h)$  must leave  $\mathcal{J}(h)$  in forward or backward time. Should an orbit belong to  $a_i^\pm(h)$ , then that orbit must be bounded in positive or negative time. Finally, any orbit which belongs to  $NT_i^\pm(h)$  cannot leave  $\mathcal{J}(h)$  through  $R_i(h)$  immediately. Thus analyzing how orbits pass through  $\mathcal{J}(h)$  can be reduced to examining the orders in which the orbits can intersect  $T_i^\pm(h)$ ,  $NT_i^\pm(h)$ , and  $a_i^\pm(h)$ .

LEMMA 4.16 (Rod, [7, Lemma 3.5 and following comments]). For  $i = 1, 2, 3$ , and  $h \in [h^-, 0]$ :

- (a) The  $T_i^\pm(h)$  are homeomorphic open disks in  $\Sigma_i$ .
- (b) The  $NT_i^\pm(h)$  are homeomorphic open sets in  $\Sigma_i$ .
- (c) The  $a_i^\pm(h)$  are homeomorphic subsets in  $\Sigma_i$ .

LEMMA 4.17 (Rod, [7, Lemma 3.6]). For  $i = 1, 2, 3$  and  $h \in [h^-, 0)$  or  $i = 3$  and  $h = 0$ :

- (a) The boundary of  $T_i^+$ ,  $\partial T_i^+$  is a continuum which separates  $\Sigma_i$ .
- (b)  $NT_i^+$  is an open annulus with boundary  $T_i \cup \gamma_i$ , where  $\gamma_i$  is a continuum which separates  $NT_i^+$  from  $T_i^+$  in  $b_i^+$ .
- (c)  $a_i^+(h) = \gamma_i(h) \cup \partial T_i^+(h)$  is a continuum which separates  $\Sigma_i$  into two components.

ASSUMPTION 10. For  $h \in [h^-, 0)$  and  $i = 1, 2, 3$ ,  $a_i^\pm$  is the intersection of a sequence of closed annuli  $A_m^\pm(i) \subset b_i^\pm$ , each containing  $a_i^\pm$  in its interior, with  $A_{m+1}^\pm(i) \subset \text{int}(A_m^\pm(i))$  for  $m = 1, 2, 3, \dots$

Assumption 10 will be used in the following section to describe the behavior of orbits passing through  $\mathcal{J}(h)$  for  $h < 0$ . Again, for conditions on  $V$  which induce the existence of such annuli, the reader is referred to [1]. To see that these conditions actually imply Assumption 10, see [3].

## 5. BOUNDED ORBITS

In this section the potential function  $V$  satisfies Assumptions 1–10 and the positive  $x_1$ -axis is taken to be gradient line. The results are presented as a classification of the solutions to (2.12) which intersect  $\mathcal{J}(h)$  for  $h \in [h^-, h^+]$ . This classification takes the form of a description of the order in which the orbits pass through  $R_i(h)$  for  $i = 1, 2, 3$ . As will be seen the classification changes dramatically depending on whether  $h > 0$ ,  $h = 0$ , or  $h < 0$ . The value of this example, however, comes from viewing  $h$  as a bifurcation parameter. In particular,  $h = 0$  is a bifurcation point where the set of bounded orbits changes from a single periodic orbit to a “pathology” of bounded orbits.

5.1.  $h \in (0, h^+]$ 

Since  $h$  is assumed fixed and greater than zero, it will be dropped from the notation.

**THEOREM 5.1.** *For  $h \in (0, h^+]$  the only bounded orbit in  $\mathcal{J}$  is  $\Pi_3$ .*

*Proof.* This theorem is actually a corollary of Theorem 3.19 for the case  $h > 0$ . Recall that in the proof an isolating neighborhood  $PN_{12}$  is constructed. In this case we can choose  $s_1 = 0$  and  $s_2 = 1$ . Hence  $\mathcal{J} = PN_{12}$ . ■

5.2.  $h = 0$ 

For this case we return to the ideas of Rod [7]; however, it will become more evident where the differences lie between the description of the bounded orbits in his example and ours. The first step is to describe the behavior of the solutions to (2.12) as they cross the disks  $D_1$ ,  $D_2$ , and  $D_3$ . Notice that  $\partial D_1 = \partial D_2 = \partial D_3$ .

**DEFINITION 5.2.** For  $i = 1, 2$ , let  $e_i(1)$  and  $e_i(2)$  be the first point and last point, respectively, at which  $\theta(z_{ii})$  intersects  $D_3$ . Let  $e_3(1) = (\tilde{x}_1, 0, -\tilde{y}_1, 0)$  where  $\tilde{x}_1$  is as in Theorem 4.4 and  $\tilde{y}_1 > 0$ . Finally, let  $e_3(2) = (\tilde{x}_1, 0, \tilde{y}_1, 0)$ .

Using Corollary 4.5 and keeping in mind the conventions used in the proof of Lemma 4.9, one can represent  $D_i$ ,  $i = 1, 2, 3$  as is done in Fig. 12.

Furthermore,  $\theta(z_{11})$  divides  $D_1$  into a closed right half disk,  $RD_1$ , and a closed left half disk  $LD_1$ . Similarly  $\theta(z_{22})$  divides  $D_2$  into  $RD_2$  and  $LD_2$ . Finally,  $D_3$  is divided into a closed upper half disk,  $UD_3$ , and a closed lower half disk,  $LD_3$ , by the orbits on the stable and unstable manifold of the origin which project onto the positive  $x_1$ -axis. In addition,  $\text{int}(RD_1)$  consists of the orbits which leave  $R_2$  and immediately enter  $R_3$ , and  $\text{int}(LD_1)$  consists of the orbits which leave  $R_3$  and immediately enter  $R_2$ .

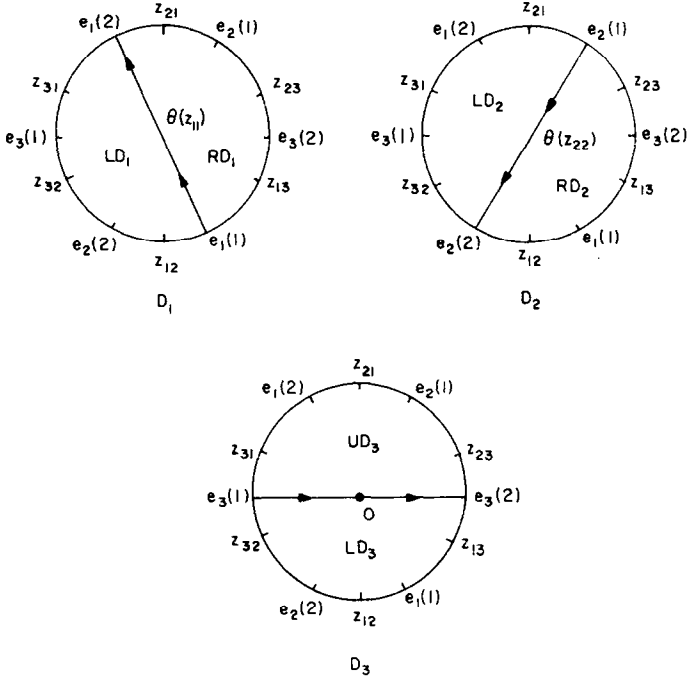


FIGURE 12

Similar statements can be made for the interiors of  $LD_2$ ,  $RD_2$ ,  $LD_3$ , and  $UD_3$ .

We are interested in describing how  $a_i^\pm$ ,  $T_i^\pm$ , and  $NT_i^\pm$  intersect  $D_j$  for  $i, j = 1, 2, 3$ . For our purposes it is sufficient to show that the geometric information present in Fig. 13 is correct. To do so we shall restrict our attention to  $RD_1$  and  $UD_3$  and claim that the arguments for the other half disks are similar. Our strategy is to determine the sets  $\partial RD_1 \cap T_i^\pm$  and  $\partial UD_3 \cap T_i^\pm$  and then use results from Section 4 to obtain Fig. 13. To do this the following notation will be useful. If  $a, b \in \partial D_1$  ( $\partial D_3$ ) then  $(a, b)$  denotes the open segment of  $\partial D_1$  ( $\partial D_3$ ) obtained by starting at  $a$  and proceeding to  $b$  along  $\partial D_1$  ( $\partial D_3$ ) in a clockwise direction.  $[a, b]$  denotes the corresponding closed segment.

We begin with two technical lemmas which, are relevant since  $D^2V(0) = -I$ .

**LEMMA 5.3.** *Let  $z = (x_0, 0, 0, x_0)$  where  $x_0 > 0$ . Let  $\tilde{z} = (x_0, 0, y_1, y_2)$  where  $y_1 < 0$ ,  $y_2 > 0$  and  $H_L(\tilde{z}) = 0$  ( $H_L$  is defined as in (2.2)). Consider  $\theta(z)$  and  $\theta(\tilde{z})$  solutions to the linear equations (2.3). Then  $\theta_1(z) \cap \theta_1(\tilde{z}) = (x_0, 0)$ .*



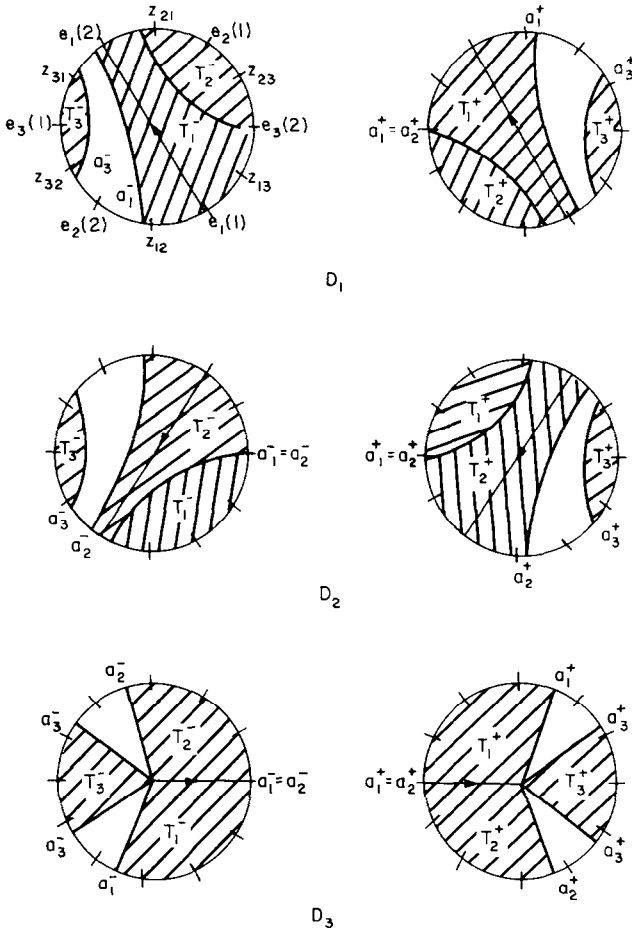


FIGURE 13

*Proof.* The exact solution to (2.3) is given by  $z(t) = \exp(tA) z(0)$ . More explicitly, we have

$$\exp(tA) = \begin{pmatrix} \theta(t) & 0 & \Psi(t) & 0 \\ 0 & \theta(t) & 0 & \Psi(t) \\ \Psi(t) & 0 & \theta(t) & 0 \\ 0 & \Psi(t) & 0 & \theta(t) \end{pmatrix},$$

where

$$\theta(t) = \cos(it) = \frac{1}{2}(e^t + e^{-t})$$

and

$$\Psi(t) = -i \sin(it) = \frac{1}{2}(e^t - e^{-t}).$$

Thus a solution to  $\theta_1(z) \cap \theta_1(\tilde{z})$  must be a solution to

$$\theta(t) x_0 = \theta(t_0) x_0 + \Psi(t_0) y_1 \quad (5.1)$$

and

$$\Psi(t) x_0 = \Psi(t_0) y_2. \quad (5.2)$$

Since  $H_L(\tilde{z}) = 0$  and  $y_1 < 0$ , one has that

$$y_1 = -\sqrt{x_0^2 + y_2^2}. \quad (5.3)$$

Substituting (5.2) and (5.3) into (5.1) gives

$$(\theta(t) - \theta(t_0)) x_0 = -\Psi(t_0) x_0 \sqrt{1 + (\Psi^2(t)/\Psi^2(t_0))}$$

or

$$[\theta(t) - \theta(t_0)]^2 = \Psi^2(t_0) + \Psi^2(t).$$

Expanding gives

$$4 - \{e^{t+t_0} + e^{t-t_0} + e^{-t+t_0} + e^{-t-t_0}\} = 0.$$

The only solution to this is  $t = t_0 = 0$ . ■

**LEMMA 5.4.** *Let  $z = (x_0, 0, -x_0, 0)$ ,  $x_0 > 0$  and  $\tilde{z} = (x_0, 0, y_1, y_2)$  where  $y_1 < 0$ ,  $y_2 > 0$  and  $H_L(\tilde{z}) = 0$ . Consider  $\theta(z)$  and  $\theta(\tilde{z})$  solutions to the linear equation (2.3). Then  $\theta_1(z) \cap \theta_1(\tilde{z}) = (x_0, 0)$ .*

*Proof.* The proof is the same as that of Proposition 5.3. ■

**PROPOSITION 5.5.** *If  $\|(\tilde{x}_1, 0)\|$  is small enough then  $z \in (e_3(1), z_{21}] \subset T_1^+$ .*

*Proof.* Choosing  $\|(\tilde{x}_1, 0)\|$  small means that one can approximate on bounded time intervals the orbits  $\theta_1(z)$  of the non-linear flow (2.18) by the orbit of  $\theta_1(z)$  of the linear flow (2.2). By Lemmas 5.3 and 5.4,  $\theta_1(z)$  (linear) is bounded by the positive  $x_1$ -axis and  $\theta_1(z_{21})$ , both of which intersect  $\Sigma_1^*(0)$ . Thus  $\theta_1(z)$  (non-linear) crosses  $\Sigma_1^*(0)$ , i.e.,  $z \in T_1^+(0)$ . ■

With these general results established, we can now turn to the specific examples. We begin with  $RD_1$ . Clearly,  $\theta(z_{11}) \subset T_1^+$ . Combining this with Proposition 5.5 gives that  $(e_3(1), z_{21}] \subset T_1^+$  and  $e_1(1) \in T_1^+$ . Lemma 4.16 says that  $T_1^+ \cap RD_1$  is an open set in  $RD_1$ ; hence, there must be a component of this open set which contains  $\theta(z_{11}) \cup [e_1(2), z_{21}]$ . Thus, we can represent  $T_1^+ \cap RD_1$  as in Fig. 13.

Since the  $x_1$ -axis is a gradient line of  $V$ ,  $e_3(2) \in T_3^+$ . Referring to the proof of Theorem 4.2 (see Fig. 8b), one can conclude that  $[z_{23}, z_{13}] \subset T_3^+$ . By definition,  $e_2(1)$  leaves  $J$  via  $\Sigma_2^*$ ; thus,  $e_2(1) \notin T_3^+ \cup T_1^+$ . Similarly,  $e_1(1) \notin T_3^+ \cup T_2^+$ . Therefore,  $\partial T_3^+ \cap (e_2(1), z_{23}) = \emptyset$  and  $\partial T_3^+ \cap (z_{13}, e_1(1)) \neq \emptyset$ . Again, this is what Fig. 13 indicates. It should be clear that the boundaries of the components of  $T_3^+$  and  $T_1^+$  shown in Fig. 13 are subsets of  $a_3^+$  and  $a_1^+$ , respectively, and that  $a_1^+ \cap a_3^+ = \emptyset$ .

To determine how  $T_i^-$  intersects  $RD_1$ , we shall consider how  $T_1^+$  and  $T_2^+$  intersect  $LD_1$  and then use the reversibility of the Hamiltonian system to obtain the desired results. We already know that  $(e_3(1), z_{21}) \subset T_1^+$  and  $e_1(1) \in T_1^+$ . A similar set of arguments shows that  $[z_{12}, e_3(1)] \subset T_2^+$ . Clearly,  $T_1^+ \cap T_2^+ = \emptyset$ ; hence,  $e_3(1) \in \partial T_1^+ \cap \partial T_2^+$ . Also, since  $e_1(1) \notin T_2^+$  we have that  $\partial T_2^+ \cap (e_1(1), z_{12}) \neq \emptyset$ .

Now consider  $UD_3$ . Let  $e$  denote a line segment connecting  $O$  and  $z_{21}$  in  $UD_3$ . Recalling the proof of the existence of  $z_{21}$  (Theorem 4.2), one recognizes that if  $z \in e \setminus \{O\}$  then  $z \in T_1^+ \cap T_2^-$ .  $[O, z_{21}] \cup e$  bounds a region in  $UD_3$  and Proposition 5.5 can be used to show that any element interior to this region is an element of  $T_1^+$ .  $[e_3(2), O]$  is defined by the  $x_1$ -axis and hence is contained in  $T_3^+$  thus by previous remarks  $[e_2(1), O] \cap T_1^+ = \emptyset$ . Therefore, we can conclude that for  $i = 1, 3$ ,  $T_i^+$  and  $a_i^+$  are as in Fig. 13. A similar argument can be used for  $T_i^-$  and  $a_i^-$ ,  $i = 2, 3$ .

We are now in a position to consider the classification scheme and prove the existence, or lack thereof, of certain orbits. Our classification will be done by describing the sequence in which the orbits pass through the interiors of the  $R_i$ 's,  $i = 1, 2, 3$ . Let  $s$  be a sequence  $\{s_k\}$  (possibly bi-infinite) where  $s_k \neq s_{k+1}$  and  $s_k \in \{-\infty, 1, 2, 3, \infty\}$ . Though we are currently concerned with  $h = 0$ , it will be shown that to classify the orbits of (2.18) which intersect  $\mathcal{J}(h)$ ,  $h \in [h^-, 0)$ , one needs the following nine types of sequences:

- (T1)  $s$  is a bi-infinite sequence  $\{s_k\}_{k=-\infty}^{\infty}$  and  $s_k \in \{1, 2, 3\}$ .
- (T2)  $s = \{s_k\}_{k=0}^{\infty}$  and  $s_k \in \{1, 2, 3\}$ .
- (T3)  $s = \{s_k\}_{k=-\infty}^0$  and  $s_k \in \{1, 2, 3\}$ .
- (T4)  $s = \{s_k\}_{k=0}^{\infty}$  and  $s_0 = -\infty$ ,  $s_k \in \{1, 2, 3\}$  for  $k \geq 1$ .
- (T5)  $s = \{s_k\}_{k=-\infty}^0$  and  $s_0 = \infty$ ,  $s_k \in \{1, 2, 3\}$  for  $k \leq -1$ .
- (T6)  $s = \{s_k\}_{k=0}^n$  and  $s_k \in \{1, 2, 3\}$ .
- (T7)  $s = \{s_k\}_{k=1}^n$  and  $s_0 = -\infty$ ,  $s_k \in \{1, 2, 3\}$  for  $1 \leq k \leq n$ .
- (T8)  $s = \{s_k\}_{k=0}^n$  and  $s_n = \infty$ ,  $s_k \in \{1, 2, 3\}$  for  $0 \leq k < n$ .
- (T9)  $s = \{s_k\}_{k=0}^n$  and  $s_0 = -\infty$ ,  $s_n = \infty$ , and  $s_k \in \{1, 2, 3\}$  for  $0 < k < n$ .

To see how these sequences describe orbits, notice that given  $s = \{s_k\}$ ,

one can associate a sequence  $R(s) = \{R(s_k)\}$ , where  $R(s_k) = R_{s_k}$  if  $s_k \in \{1, 2, 3\}$ . One says that an orbit follows  $s$ , if as time increases the orbit passes into the interior of each  $R_{s_k}$  successively. If  $s_0 = -\infty$  as in (T4), (T7), and (T9), then one says that the orbit entered  $R(s_1)$  via  $\Sigma_{s_1}^*$ . Similarly, if  $s_n = \infty$  as in (T5), (T8), and (T9), then one says that the orbit left  $R(s_{n-1})$  via  $\Sigma_{s_{n-1}}^*$ .

Though all sequences of the form (T1)–(T9) are necessary in the case  $h < 0$ , for  $h = 0$  the results are much simpler. This is what one expects since  $h = 0$  is the bifurcation point. As Theorem 5.6 demonstrates, the only sequences which appear fall into the types (T6)–(T9), and furthermore, most sequences  $s$  in (T6)–(T9) are not realized. However, comparing the results of Theorem 5.6 with those of Theorem 5.8 allows one to see how the appearance of the critical point at the origin “separates” orbits entering and exiting  $J$  via  $\Sigma_1 \cup \Sigma_2$ . This separation is the bifurcation which gives rise to the pathology of orbits in  $\mathcal{J}$ .

**THEOREM 5.6.**  *$s$  is a sequence representing an orbit on the energy level  $H = 0$  if and only if  $s = \{s_k\}$  satisfies:*

- (1) *There exists at most one  $k$  such that  $s_k = 3$ .*
- (2) *If  $\{s_1, s_2, s_3\} = \{1, 2, 1\}$  or  $\{2, 1, 2\}$  then  $s_0 = -\infty$  and  $s_4 = \infty$ .*
- (3) *If  $\{s_1, s_2, s_3\} = \{1, 2, 3\}$  or  $\{2, 1, 3\}$  then  $s_0 = -\infty$ .*
- (4) *If  $\{s_1, s_2, s_3\} = \{3, 2, 1\}$  or  $\{3, 1, 2\}$  then  $s_4 = \infty$ .*
- (5)  *$\{1, 2\}$  and  $\{2, 1\}$  are not possible sequences.*

*Proof.* We first show that if  $s$  represents an orbit then  $s$  satisfies (1) through (5). Let  $\theta(z)$  be an orbit represented by  $s$ .

(1) We need to show that if  $s_l = 3$  and  $k \neq l$  then  $s_k \neq 3$ . Without loss of generality we can assume  $s_0 = 3$  and we need only show that  $s_k \neq 3$  for all  $k > 0$ . If  $s_1 = \infty$  or  $s_1$  does not exist then we are done. Thus  $s_1 = 1$  or  $s_1 = 2$ . In either case the argument that follows is similar so let  $s_1 = 2$ . This implies there exists  $t_0$  such that  $z \cdot t_0 \in LD_1$ . By Fig. 13, one can conclude that  $z \cdot t_0 \in T_1^+$ ,  $z \cdot t_0 \in T_2^+$  or  $z \cdot t_0 \in a_1^+ = a_2^+$ . In the first case, by the definition of  $T_1^+$ ,  $s_2 = 1$  and  $s_3 = \infty$ . In the second case  $s_2 = \infty$  and finally if  $z \cdot t_0 \in a_1^+ = a_2^+$  then  $s_2$  does not exist.

(2) The assumption that  $\{1, 2, 1\} = \{s_1, s_2, s_3\}$  implies that there exists  $t_0 < t_1$  such that  $z \cdot t_0 \in LD_3$ ,  $z \cdot t_1 \in UD_3$  and  $z \cdot (t_0, t_1) \subset R_2$ . Referring to Fig. 13 we have that  $z \cdot t_1 \notin T_2^-$  implies  $z \cdot t_1 \in T_1^+$ , and  $z \cdot t_0 \notin T_2^+$  implies  $z \cdot t_0 \in T_1^-$ ; thus  $s = \{-\infty, 1, 2, 1, \infty\}$ . A similar argument applies to  $\{2, 1, 2\}$ .

(3) Assume  $\{s_1, s_2, s_3\} = \{1, 2, 3\}$ , then there exists  $t_0 < t_1$  such that  $z \cdot t_0 \in LD_3$ ,  $z \cdot t_1 \in RD_1$  and  $z \cdot (t_0, t_1) \subset R_2$ . Since  $s_3 = 3$ ,  $z \cdot t_0 \notin T_2^+$  hence

$z \cdot t_0 \in T_1^-$  and therefore  $s_0 = -\infty$ . If  $\{s_1, s_2, s_3\} = \{2, 1, 3\}$  the argument is similar.

(4) The argument is similar to that of (3).

(5)  $s = \{1, 2\}$  means that  $a_1^-$  intersects  $a_2^+$  non-trivially in  $LD_3 \setminus \{O\}$ . Figure 13 says this cannot happen.

Showing that, if  $s$  satisfies (1)–(5) then there exists an orbit whose path is represented by  $s$ , is easy but tedious. (1)–(4) implies that the length of  $s$  is less than or equal to 7. Thus there exists a finite number of possible orbit types. Checking that each orbit in fact exists is therefore possible and left to the reader. As an example we shall demonstrate the existence of  $\theta(z)$  given a particular  $s$ .

Let  $s = \{-\infty, 1, 2, 3, 1, 2, \infty\} = \{s_k\}_{k=0}^6$ . We need to find  $z$  such that  $\theta(z)$  is represented by  $s$ . Let  $K_0$  be a closed set with non-empty interior such that  $K_0 \subset LD_3 \setminus (T_3^+ \cup T_2^+ \cup a_2^+)$  and  $K_0 \cap a_3^+ \neq \emptyset$ . We can choose  $K_0$  small enough so that  $\varphi_2^+(K_0) \subset RD_1$ . Thus if  $z' \in K_0$  and  $s'$  represents  $\theta(z')$  then  $s' = \{-\infty, 1, 2, 3, \dots\}$ . There exist subsets of  $K_0$  which in forward time pass near  $\Pi_3(0)$  and exit from  $R_3$  into  $R_1$ . (See Rod [7] for details). Thus  $\phi \neq K_1 \equiv \varphi_3^+ \circ \varphi_2^+(K_0) \cap D_2$ . Furthermore,  $K_1 \cap T_2^+ \neq \emptyset$ . Thus if  $z \in K_1$  and  $s$  represents  $\theta(z)$  then  $s = \{-\infty, 1, 2, 3, 1, 2, \infty\}$ .

**COROLLARY 5.7.** (a) *If  $s = \{3, 1\}$ ,  $\{1, 3\}$ ,  $\{2, 3\}$ , or  $\{3, 2\}$  then there exists at least one orbit  $\theta(z)$  which is represented by  $s$ .*

(b) *There exists at least countably many distinct orbits of the type  $s = \{1, 3, 1\}$ ,  $\{2, 3, 2\}$ ,  $\{1, 3, 2\}$ , or  $\{2, 3, 1\}$ .*

(c) *There exists uncountably many distinct orbits of the type  $s$  if  $s$  is not of type (a) or (b).*

*Proof.* See Rod [7] for details. ■

### 5.3. $h \in [h^-, 0)$

In this region we have three periodic orbits whose stable and unstable manifolds intersect transversally. For this case the work has been done for us by Churchill and Rod [2, 3]. Theorem 5.8 implies that the set of bounded solutions is much more complicated when  $h < 0$  than when  $h = 0$ , since any sequence of type (T1)–(T9) corresponds to an orbit passing through  $\mathcal{J}$ .

**THEOREM 5.8.** (a) *Let  $s$  be of type (T1). Then there exists uncountably many solutions of (2.18) which pass through the sequence of regions  $R(s)$ .*

(b) *Let  $s$  be of the type (T2)–(T5) or (T7)–(T9). Then there exist uncountably many solutions passing through the sequence  $R(s)$ .*

(c) *Let  $s$  be of type (T6) with  $n \geq 3$ . Then there exist at least countably many solutions passing through  $R(s)$ .*

*Proof.* This theorem follows from a collection of theorems in [2, 3]. More specifically, (a) follows from Theorem 1.3 in [3] and (b) and (c) follow from Theorem 6.4 in [2]. ■

**THEOREM 5.9.** *Assume  $s$  is of type (T1) and periodic with period  $n$ , i.e.,  $s_k = s_{k+n}$  for all  $k$  where  $n > 0$ . Then given any  $m > 0$ , there exists at least one periodic orbit which passes through the sequence*

$$\{s_k\}_{k=k_0}^{k_0+n}$$

*$m$ -times and then closes up.*

*Proof.* See [3, Theorem 1.3]. ■

Both Theorem 5.8 and Theorem 5.9 are dependent on Theorem 1.3 of Churchill and Rod [3]. In their paper they give six hypotheses that must be satisfied in order for the theorem to hold. It is straightforward to check that the first two are satisfied. The third hypothesis follows by arguments similar to those of [3, Section 3]. The fourth hypothesis is Assumption 10 in Section 4 of this paper. The fifth hypothesis is also satisfied by the results of Section 4. Hence only their sixth hypothesis needs to be demonstrated. This is the content of the following proposition.

**PROPOSITION 5.10.** *Let  $U \subset T_i^-$  be the maximal connected open (relative to  $\Sigma_i$ ) set containing  $z_{ij} \in T_i^-$  which is carried homeomorphically by the flow onto  $U^* \subset T_j^+$ . Also, let  $K \subset a_i^-$  be any connected set intersecting the closure of  $U$  which is carried homeomorphically by the flow onto  $K^* \subset T_j^+ \cup a_j^+$ . Then:*

(a) *The closure of  $U$  is carried homeomorphically by the flow onto the closure of  $U^*$ .*

(b) *The closure of  $K$  is carried homeomorphically by the flow onto the closure of  $K^*$ .*

*Proof.* Let  $\varphi: U \rightarrow U^*$  denote the homeomorphism. If  $p \in U$  then  $\varphi(p) = q \in U^*$  and there exists  $t(p) \geq 0$  so that  $p \cdot t(p) = q$ . Let  $\tilde{q}$  be in the closure of  $U^*$  and let  $q_n \rightarrow \tilde{q}$  where  $q_n \in U^*$  for all  $n$ . Then there exists  $p_n \in U$  and  $t(p_n) \geq 0$  such that  $p_n \cdot t(p_n) = q_n$ . The closure of  $U$  is compact, hence there exists a convergent subsequence  $\{p_m\}$  such that  $p_m \rightarrow \tilde{p}$  an element of the closure of  $U$ . If it can be shown that there exists a  $t(\tilde{p})$  such that  $\tilde{p} \cdot t(\tilde{p}) = \tilde{q}$ , then (a) will be shown to be true. It is sufficient to show that  $\{t(p_n)\}$  is bounded.

So consider the case  $i = 1, j = 2$ . Define  $M(z) = (m_1, m_2, m_3)$  by letting  $m_i$  equal the number of times  $\theta_1(z)$  crosses  $D_i$  for  $i = 1, 2$  and  $m_3$  equal the number of times  $\theta_1(z)$  crosses the positive  $x_1$ -axis. Since for  $z \in U$ ,  $\theta_1(z)$

always crosses each arc transversally,  $M: U \rightarrow Z \times Z \times Z$  is continuous. But  $M(z_{12}) = (0, 0, 1)$ ; thus  $M(z) = (0, 0, 1)$  for all  $z \in U$ . If  $\{t(p_n)\}$  is unbounded then either:

- (a) there exists  $p_n \in U$  such that  $M(p_n) = (m_1, m_2, m_3)$ , where either  $m_3 > 1$  or  $m_i > 0$  for  $i = 1, 2$ , or
- (b)  $\theta(p_n)$  remains arbitrarily long in  $R_3$ .

Case (a) cannot happen since  $M$  is continuous and case (b) cannot happen since this would force  $m_3$  to be arbitrarily large.

The proof for the other  $z_{ij}$ 's follows in a similar manner. The proof of (b) is also similar. ■

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